SUPERSYMMETRY

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Contents

Lecture 1.

Tuesday, January 22, 2019

We begin with admin notes, as usual. The instructor for this course is David Skinner (http://www.damtp.cam.ac.uk/people/dbs26/). The main text of this course will be Mirror Symmetry (ed. Vafa), the PDF of which is available for free here: https://www.claymath.org/library/monographs/cmim01c.pdf.

What is SUSY and why do we care? In any quantum theory involving fermions, we can divide the Hilbert space into a bosonic and a fermionic part,

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F, \tag{1.1}$$

where the bosonic part includes an even number of fermionic excitations and the fermionic part has an odd number of fermionic excitations.

Definition 1.2. A theory is *supersymmetric* if $\exists$ a fermionic operator Q which maps between the bosonic and fermionic parts of the Hilbert space,

$$Q : \mathcal{H}_B \to \mathcal{H}_F, \mathcal{H}_F \to \mathcal{H}_B,$$

such that

$$\{Q, Q^\dagger\} = 2H, \quad Q^2 = (Q^\dagger)^2 = 0. \tag{1.3}$$

where H is the Hamiltonian of the theory and $Q^\dagger$ represents the adjoint of Q with respect to the inner product on $\mathcal{H}$.

This algebra has some immediate consequences:

- $2[H, Q] = \left[\{Q, Q^\dagger\}, Q\right] = (QQ^\dagger + Q^\dagger Q)Q - Q(QQ^\dagger + Q^\dagger Q)$. But the Q^2 terms are zero by definition and the $QQ^\dagger Q$ terms cancel. Therefore $[H, Q] = 0 \implies Q$ is conserved, and the transformations it generates will be symmetries. These symmetries are parametrized by fermionic parameters, and are called *supersymmetries*. Q is known as the *supercharge*.
- For any state $|\psi\rangle \in \mathcal{H}$, the expectation value of the Hamiltonian in this state, $\langle\psi|H|\psi\rangle$, is given by

$$\begin{aligned}\langle\psi|H|\psi\rangle &= \langle\psi|QQ^\dagger + Q^\dagger Q|\psi\rangle \\ &= ||Q^\dagger|\psi\rangle||^2 + ||Q|\psi\rangle||^2 \geq 0.\end{aligned}$$

  Therefore all states have non-negative energy, with equality iff the ground state $|\Omega\rangle$ obeys $Q|\Omega\rangle = Q^\dagger|\Omega\rangle = 0$. That is, $|\Omega\rangle$ has $E = 0$ iff it is supersymmetric.

Why is such a theory interesting? There are a few reasons to care.

- Phenomenology– in the Standard Model, the coupling strength of different forces depends on the energy scale we are interested in. The EM coupling constant α and the equivalent for QCD meet at some energy scale, and the weak force meets these couplings at some other coupling scale. Based on only SM particles, the unification scales are different. But adding SUSY particles could allow for all three forces to unify at a single energy scale.
- Matter in the Standard Model transforms in representations of $SO(1) \subset SU(5) \subset SU(3) \times SU(2) \times U(1)$.
- Previously, it was thought that SUSY could address the "hierarchy problem," i.e. why quantum loop corrections don't conspire to make the Higgs mass incredibly large, rather than the value it was discovered at ($\sim 125\,\text{GeV}$). However, the LHC has found no evidence for supersymmetric particles in most of the configurations that would solve this problem.
- Most importantly for this course, SUSY helps us to better understand QFT. In quantum mechanics, we learned about some toy models like the harmonic oscillator, the infinite square well, and the hydrogen atom. These toy models were necessarily simplified and exhibited a lot of symmetry– it was only later that we introduced perturbative methods to find approximate solutions to more complicated (and more realistic) problems. But in QFT, the only exactly solvable model we've seen so far is the free theory. SUSY will provide us with other QFTs in which we can compute some quantities exactly. Moreover, these quantities often reveal connections between QFT, geometry, and topology (cf. mirror symmetry).

This course will be more focused on the mathematical structure of supersymmetric theories rather than the phenomenological concerns of how we might observe e.g. superpartners at colliders like the LHC.

Path integrals in QFT The path integral formalism in QFT centers around an expression which looks like

$$\int_{\mathcal{C}} e^{-S[X]/\hbar} \mathcal{D}X,$$

where X is some field and $\mathcal{C}$ is the space of all field configurations. We have written this path integral in Euclidean signature; otherwise, the exponent would be $-iS[X]/\hbar$.

However, there are some problems with this setup. This integral is not well-defined (what is the integration measure $\mathcal{D}X$?) and is formidably hard to compute. As a toy model, suppose the whole universe is a single point, $M =$ point (zero-dimensional QFT, if you like). Then a field on M could be $X : \text{pt} \to \mathbb{R}$ and the path integral becomes

$$\int_{\mathbb{R}} e^{-S[X]/\hbar} dX, \tag{1.4}$$

where dX is now just the ordinary integration measure on the real line. To compute this integral, we also need an action. Suppose we had an action of the form

$$S[X] = \frac{mX^2}{2} + \frac{\lambda}{6} + \frac{gX^6}{6!}. \tag{1.5}$$

Since there is only one point in our spacetime, we have no way to take derivatives and therefore no kinetic term in our action. But it turns out that even with this relatively simple-looking action, this integral

$$Z = \int_{\mathbb{R}} e^{-S[X]/\hbar} dX$$

is still hard to do. As $\hbar \to 0$ (the semi-classical limit), we can obtain an asymptotic series. If $S[X]$ has an isolated minimum at some $X_0 \in \mathbb{R}$ (that is, $\partial_X S[X] = 0, \partial_X^2 S[X] > 0$), then we can apply a steepest-descent approach and say that the integral will be dominated by its value at the minimum $S[X_0]$, plus some higher-order corrections. Thus

$$Z \sim_{\hbar\to 0} \frac{e^{-S[x_0]/\hbar}}{\sqrt{\frac{\partial^2 S}{\partial X^2}(x_0)}} \Big(1 + A\hbar + B\hbar^2 + \dots\Big), \tag{1.6}$$

where the numerator $e^{-S[X_0]/\hbar}$ represents tree-level Feynman diagrams, the denominator represents one-loop diagrams, and the asymptotic series represents high-energy corrections. However, we should be careful about the radius of convergence of this series. If the asymptotic series $\Big(1 + A\hbar + B\hbar^2 + \dots\Big)$ converges for $\hbar > 0$, it certainly must converge under sending $\hbar \to -\hbar$.[1] But going back to our path integral, if $\hbar < 0$, then our exponential factor $e^{-S[X]/\hbar}$ blows up for large actions $S[X]$, and so this integral has no chance of converging for any $\hbar < 0$, which means that our asymptotic series cannot converge as a Taylor series.

SUSY in $d = 0$ Let us introduce some quantities we will call Grassman variables.

Definition 1.7. *Grassman variables* are a set of n elements ψ^a obeying the algebra

$$\psi^a \psi^b = -\psi^b \psi^a. \tag{1.8}$$

In particular, note that $(\psi^a)^2 = 0$.

These elements might look familiar– for the fermionic spinor field in QFT, we had equal time anti-commutation relations, $\{\psi^\alpha(\mathbf{x}), \psi^\beta(\mathbf{y})\} = 0$, and similarly in general relativity we saw that the appropriate multiplication law on one-forms was the wedge product, $dx^a \wedge dx^b = -dx^b \wedge dx^a$. These similarities are not a coincidence, but it will take us some time to unravel the implications.

Now consider some operator $F(\psi)$ which is polynomial in our Grassman variables ψ^a. We can write it as

$$F(\psi) = f + \rho_a \psi^a + \phi_{ab} \psi^a \psi^b + \dots + g_{a_1 \dots a_n} \psi^{a_1} \dots \psi^{a_n}. \tag{1.9}$$

However, note that this expansion *must* terminate! Since $(\psi^a)^2 = 0$, once we try to write down a term like $\psi^{a_1} \dots \psi^{a_{n+1}}$, we run out of distinct ψ^as to multiply together (there are only n of them) and as soon as we get a $(\psi^a)^2$, the whole term goes to zero. n.b. since $\psi^a \psi^b$ is antisymmetric by definition, we may as well take its coefficient ϕ_{ab} to be antisymmetric as well, $\phi_{ab} = -\phi_{ba}$.

If $F(\psi)$ is bosonic (commuting), then f, ϕ, and the other even coefficients must also be bosonic, whereas ρ_a and the other odd-ψ coefficients must be fermionic. For strings of Grassman variables, we define derivatives by

$$\frac{\partial}{\partial \psi^a}(\psi^b \dots) = \delta^b{}_a(\dots) - \psi^b \frac{\partial}{\partial \psi^a}(\dots), \tag{1.10}$$

so that derivatives must also anticommute with the Grassman variables in our version of the Leibniz rule.

[1] Strictly, this is because the function must converge within some disc in the complex plane. I believe this is also what allows us to perform Wick rotations when calculating path integrals.

Lecture 2.

Thursday, January 24, 2019

Last time, we introduced the Grassman variables. They are a set of elements which anticommute and obey a variation of the Leibniz rule,

$$\frac{\partial}{\partial \psi^a}(\psi^b \dots) = \delta^b{}_a(\dots) - \psi^b \frac{\partial}{\partial \psi^a}(\dots).$$

Of course, now that we've defined differentiation we'd naturally like to define integration as well. Since $(\psi)^2 = 0$, we only need to define

$$\int 1\, d\psi \text{ and } \int \psi d\psi.$$

We want our integral to be "translation-invariant," i.e.

$$\int (\psi + \eta) d\psi = \int \psi d\eta \implies \int 1\, d\psi = 0 \tag{2.1}$$

for $\eta \in \mathbb{R}$. We then normalize by choosing

$$\int \psi d\psi := 1, \tag{2.2}$$

known as *Berezin integration*. Suppose we have n fermions $\psi^1, \dots, \psi^n$, with

$$\int \psi^1 \psi^2 \dots \psi^2 \underbrace{d\psi^n d\psi^{n-1} \dots d\psi^1}_{d^n\psi} = 1. \tag{2.3}$$

We must have the $d\psi$s in this order in order to perform each of the integrals, so that

$$\int \psi^{a_1} \dots \psi^{a_n} d^n\psi = \epsilon^{a_1 a_2 \dots a_n}, \tag{2.4}$$

with ϵ the totally antisymmetric ϵ-symbol.

Now let

$$\psi'^a = N^a{}_b \psi^b \text{ for } N \in GL(n). \tag{2.5}$$

We have

$$\int \psi'^a \psi'^b \dots \psi'^d d^n\psi = N^a{}_e N^b{}_f \dots N^d{}_g \int \psi^e \psi^f \dots \psi^g d^n\psi, \tag{2.6}$$

where we have brought the N ($n \times n$ matrices) by the linearity of the integral– their entries are just numbers). But indeed we can perform the integral now– it is

$$\begin{aligned}\int \psi'^a \psi'^b \dots \psi'^d d^n\psi &= N^a{}_e N^b{}_f \dots N^d{}_g \epsilon^{ef\dots g} \\ &= \det(N) \varepsilon^{ab\dots d} \\ &= \det(N) \int \psi'^a \psi'^b \dots \psi'^d d^n\psi'.\end{aligned}$$

Comparing, we see that if $\psi'^a = N^a{}_b \psi^b$, then

$$d^n\psi' = \frac{1}{\det(N)} d^n\psi, \tag{2.7}$$

which is the opposite of the usual convention.

Example 2.8. If we have $\chi = a\psi$, then

$$\int \chi d\chi = 1 = a \int \psi d\chi \implies d\chi = \frac{d\psi}{a}, \tag{2.9}$$

recalling that $\int \psi d\psi = 1$.

For QFT, we often need Gaussian integrals. Suppose ψ^1, ψ^2 are fermionic and let

$$S(\psi) = \frac{1}{2}\psi^1 M \psi^2, \tag{2.10}$$

some sort of action in terms of the fermionic fields ψ^1, ψ^2. There are no kinetic terms since we're still working in zero dimensions. Then an integral we might like to calculate is

$$\int e^{-S(\psi^a)} d\psi^1 d\psi^2. \tag{2.11}$$

But in fact, this integral will be dead simple to calculate. If we Taylor expand the exponential, the expansion actually terminates at the first non-trivial term since the order $(\psi^1 M \psi^2)^2$ term would contain a $(\psi^1)^2$, which vanishes.

Therefore our integral becomes

$$\int e^{-S(\psi^a)} d\psi^1 d\psi^2 = \int \left((1 - \frac{1}{2}\psi^1 M \psi^2 \right) d\psi^1 d\psi^2 = \frac{1}{2} M. \tag{2.12}$$

More generally, for $2m$ fermions with "action"

$$S(\psi^a) = \frac{1}{2}\psi^a M_{ab} \psi^b, \tag{2.13}$$

where we shall take $M_{ab} = -M_{ba}$ to be antisymmetric WLOG, our action integral becomes

$$\begin{aligned}
\int e^{-S(\psi)} d^{2m}\psi &= \int \sum_{k=0}^{\psi} \frac{(-1)^k}{k!} \frac{1}{2^k} \left(\psi^a M_{ab} \psi^b\right)^k d^{2m}\psi \\
&= \frac{(-1)^k}{2^m m!} \int \left(\psi^a M_{ab} \psi^b\right)^m d^{2m}\psi \\
&= \frac{(-1)^m}{2^m m!} \epsilon^{a_1 b_1 \ldots a_m b_m} M_{a_1 b_1} M_{a_2 b_2} \ldots M_{a_m b_m} \\
&= \sqrt{\det M},
\end{aligned}$$

sometimes called the Pfaffian of the matrix M. (For "bosons," we would have instead $\int e^{-\frac{1}{2}x^a M_{ab} x^b} d^{2m}x = \frac{(2\pi)^m}{\sqrt{\det M}}$.)

Supersymmetric integrals and localization Consider a $d = 0$ theory of one bosonic variable x and two fermions ψ^1, ψ^2. We certainly need at least two fermions in order to have something quadratic in the fermions that is non-vanishing. Take

$$S(x, \psi^i) = V(x) - \psi^a \psi^2 U(x) \tag{2.14}$$

as our action. Our V captures some sort of interactions between bosons in our theory, and any nontrivial terms in U will likewise result in some sort of interactions between the fermions and the boson. We see that even in $d = 0$, for generic V, U the integral

$$\int e^{-S(x,\psi^i)} dx d\psi^1 d\psi^2$$

is difficult.

Let's specialize and see if there's a case we can solve. Suppose we choose a polynomial $W(x)$ and take

$$S(x, \psi^i) = \frac{1}{2}(\partial W)^2 - \bar{\psi}\psi \partial^2 W \tag{2.15}$$

where $\psi = \psi_1 + i\psi_2, \bar{\psi} = \psi_1 - i\psi_2$. Derivatives are clearly taken with respect to x. What we've done is constructed a specific relation between the two terms in the action.

Now we observe that this action $S(x, \psi, \bar{\psi})$ is invariant under

$$\begin{aligned}
\delta x &= \epsilon\psi - \bar{\epsilon}\bar{\psi} \\
\delta\psi &= \bar{\epsilon}\partial W \\
\delta\bar{\psi} &= -\epsilon\partial W,
\end{aligned}$$

where $\epsilon, \bar{\epsilon}$ are fermionic parameters. This gives us variations of the right type (e.g. $\epsilon\psi$ is bosonic).

Let us check the variation of the action. We'll just check the ϵ terms– the $\bar{\epsilon}$ terms are similar.

$$\delta_\epsilon S = \partial W \partial^2 W \epsilon\psi - \epsilon \partial W \psi \partial^2 W - \bar{\psi}\psi(\epsilon\psi\partial^3 W),$$

where the last term comes from taking the chain rule since W depends on x which has some variation. But these first two terms clearly cancel (ϵ and W are just numbers, so they commute with fields) and the last term is zero because we have a ψ^2.

Since we have a symmetry of the action, we get some charges. We write $\delta = \epsilon Q + \bar{\epsilon}\bar{Q}$, where $Q, \bar{Q}$ are called *supercharges*, and

$$\begin{aligned} Qx &= \psi \quad \bar{Q}x = -\bar{\psi} \\ Q\psi &= 0 \quad \bar{Q}\psi = \partial W \\ Q\bar{\psi} &= \partial W \quad \bar{Q}\bar{\psi} = 0. \end{aligned}$$

We may write

$$\begin{aligned} Q &= \psi \frac{\partial}{\partial x} + \partial W \frac{\partial}{\partial \bar{\psi}} \\ \bar{Q} &= -\bar{\psi}\frac{\partial}{\partial x} + \partial W \frac{\partial}{\partial \psi}. \end{aligned}$$

These generators obey $\{Q, \bar{Q}\} = 0$. Note that there is no Hamiltonian H since the Hamiltonian is the generator of time translations and we are still in $d = 0$.

Let's observe now that the supersymmetric "path" integral $\int e^{-S(x,\psi,\bar{\psi})} dx d\psi d\bar{\psi}$ is in fact really easy to compute. Suppose we rescale $W \to \lambda W, \lambda \in \mathbb{R}_+$ both in the action, $S \to S_\lambda$ and in the SUSY transformation, $Q \to Q_\lambda, \bar{Q} \to \bar{Q}_\lambda$ (replacing W with λW everywhere).

Now we have an action which appears to be parametrized by λ,

$$I(\lambda) = \int e^{-S_\lambda(x,\psi,\bar{\psi})} dx d^2\psi. \tag{2.16}$$

But note that this in fact obeys $\frac{dI}{d\lambda} = 0$, and is therefore independent of λ.

Proof.

$$\begin{aligned} \frac{dI}{d\lambda} &= \int \frac{\partial}{\partial \lambda} e^{-S_\lambda} dx d^2\psi \\ &= -\int \Big(\lambda(\partial W)^2 - \bar{\psi}\psi\partial^2 W)\Big) e^{-S_\lambda} dx d^2\psi \\ &= -\int \bar{Q}_\lambda(\partial W \psi) e^{-S_\lambda} dx d^2\psi \\ &= -\int \bar{Q}_\lambda(\partial W \psi e^{-S_\lambda}) dx d^2\psi. \end{aligned}$$

But since $\bar{Q}_\lambda = -\bar{\psi}\frac{\partial}{\partial x} + (\lambda \partial W)\frac{\partial}{\partial \psi}$, this vanishes. The entire term in the parentheses is at most linear in ψ, so after taking the ∂_ψ derivative in $\bar{Q}$, we have the integral of something constant in ψ with respect to $d^2\psi$, which is zero. The ∂_x term vanishes because what remains is a total derivative of something being evaluated at the boundaries. ☒

We conclude that

$$I(1) = \lim_{\lambda \to \infty} I(\lambda), \tag{2.17}$$

which means that as $\lambda \to \infty$, the $e^{-\frac{\lambda^2}{2}(\partial W)^2}$ term suppresses the action integral everywhere except where $\partial W = 0$. Thus the integral *localizes* to critical points of $W(x)$.

Lecture 3.

Tuesday, January 29, 2019

Last time, we wrote down a particular action for our (zero-dimensional) theory:

$$S(x,\psi^i)=\frac{1}{2}(\partial W)^2-\bar{\psi}\psi\partial^2 W,$$

with $W(x)$ a polynomial. What we found via a scaling argument was that the integral

$$I=\int e^{-S(x,\psi,\bar{\psi}}dxd^2\psi$$

in fact localizes to the critical points of $W(x)$.

Now suppose we have a group G acting freely on our space of fields $\mathcal{C}$, and suppose the action and integration measure are G-invariant. For example,

$$\int_{\mathbb{R}^2\setminus\{0\}}e^{-S(x,y)}dxdy$$

with $G=SO(2)$ and S just a function of $r=\sqrt{x^2+y^2}$. In this case, we would recognize that by changing to polar coordinates, we can make the angular integral trivial and just worry about an integral over dr.

More generally, we should decompose our integration domain $\mathcal{C}$ into the orbits of G, $G\times\mathcal{C}/G$, and then integrate over G to obtain $\mathrm{vol}(G)$. However, if G is a fermionic group, then $\mathrm{vol}(G)=0$ since $0\int_G 1d^{\dim G}\theta$. More generally, if $G:\mathcal{C}\to\mathcal{C}$ has some fixed points we can only get contributions to the integral from neighborhoods of these fixed points.

In our case, we have

$$\delta\psi=\bar{\epsilon}\partial W,\quad \delta\bar{\psi}=-\epsilon\partial W, \tag{3.1}$$

so fixed points of our SUSY theory are critical points of $W(x)$. Away from such critical points, let us define some new fields

$$y=x-\frac{\bar{\psi}\psi}{\partial W},\quad \chi=\psi\sqrt{\partial W},\quad \bar{\chi}=\bar{\psi}. \tag{3.2}$$

Exercise 3.3. Show that $dxd^2\psi=\sqrt{\partial W(y)}dyd^2\chi$, where W is considered as a function of y.

If we work this out, we find that

$$\delta y=\epsilon\psi-\bar{\epsilon}\bar{\psi}-\frac{\epsilon\partial W\psi}{\partial W}+\frac{\bar{\psi}\bar{\epsilon}\partial W}{\partial W}=0 \tag{3.4}$$

away from critical points. Thus $S(y,0,0)=\frac{1}{2}(\partial W(y))^2-\frac{1}{2}(\partial W(x))^2-\partial W\partial^2 W\frac{\bar{\psi}\psi}{\partial W}=S(x,\psi,\bar{\psi})$. We conclude that

$$\int_{U^C}e^{-S(x,\psi,\bar{\psi})}dxd^2\psi=\int e^{-S(y,0,0)}\sqrt{\partial W(y)}dyd^2\chi=0, \tag{3.5}$$

where U is an open neighborhood of $\{\partial W=0\}$ with $U^C=\mathcal{C}\setminus U$ the complement in $\mathcal{C}$. This is a different way of seeing what we computed last time– the integral localizes to (a neighborhood of) fixed points of SUSY transformations.

Near any isolated critical point x_*, we have $W(x)=W(x_*)+\frac{c_*}{2}(x-x_*)^2+\ldots$, so our action becomes

$$S^{(2)}(x,\psi,\bar{\psi})=\frac{c_*^2}{2}(x-x_*)^2+\bar{\psi}\psi c_*. \tag{3.6}$$

Hence

$$\begin{aligned}
I&=\int e^{-S(x,\psi,\bar{\psi})}\frac{dx}{\sqrt{2\pi}}d^2\psi\\
&=\int e^{-\frac{c_*^2}{2}(x-x_*)^2}(-1+\bar{\psi}\psi c_*)dxd^2\psi\\
&=\frac{c_*}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{c_*^2}{2}(x-x_*)^2}dx\\
&=\frac{c_*}{|c_*|}=\pm 1.
\end{aligned}$$

If W has several critical points, $I = \sum_{c_*|\partial W(c_*)=0} \frac{c_*}{|c_*|}$.

This is a remarkably simplifying fact. This tells us that for each local maximum of W, we get -1 and for each local minimum, we get -1. Thus

- $I = 0$ if W is an odd degree polynomial
- $I = -1$ if W is an even degree polynomial and $W \to -\infty$ as $|x| \to \infty$
- $I = +1$ if W is an even degree polynomial and $W \to +\infty$ as $|x| \to \infty$.

Whereas we might have thought that this integral a priori could have been arbitrarily hard to compute and depend on the form of W in some complicated way, it turns out that the integral takes only three discrete values and is determined by some sort of topological property of W.

Landau-Ginzburg theory Let's do one more example in $d = 0$. Consider a complex bosonic variable $z \in \mathbb{C}$ and two complex fermions ψ_1, ψ_2. Choose holomorphic $W(z)$ with an action

$$S(z, \psi_1, \psi_2) = |\partial W|^2 + \partial^2 W \psi_1 \psi_2 - \overline{\partial^2 W} \bar{\psi}_1 \psi_2. \tag{3.7}$$

We claim this is invariant under

$$\begin{aligned} \delta z = \epsilon_1 \psi_1 + \epsilon_2 \psi_2, &\quad \bar{\delta} \bar{z} = \bar{\epsilon}_1 \bar{\psi}_1 + \bar{\epsilon}_2 \bar{\psi}_2, \\ \delta \psi_1 = \epsilon_2 \overline{\partial W}, &\quad \bar{\delta} \bar{\psi}_1 = \bar{\epsilon}_2 \partial W, \\ \delta \psi_2 = -\epsilon_1 \overline{\partial W}, &\quad \bar{\delta} \bar{\psi}_2 = \bar{\epsilon}_1 \partial W. \end{aligned}$$

We also have $\bar{\delta} z = \bar{\delta} \psi_i = 0, \delta \bar{z} = \delta \bar{\psi}_i = 0$.

One can now check that our SUSY operators satisfy

$$\{Q_i, \bar{Q}_j\} = 0,$$

but $\{Q_i, Q_j\} = 0 = \{\bar{Q}_i, \bar{Q}_j\}$ hold only "on-shell," i.e. for $\partial^2 W = 0 = \overline{\partial^2 W}$. Again, by rescaling $W \to \lambda W$ for $\lambda \in \mathbb{R}_+$, we can localize our integral to critical points of $W(z)$, where

$$W(z) \approx W(z_*) + \frac{\alpha_*}{2}(z - z_*)^2 + \dots \tag{3.8}$$

and our integral therefore becomes

$$S^{(2)}(z, \psi_i) \simeq |\alpha_*|^2 |z - z_*|^2 + \alpha_* \psi_1 \psi_2 - \bar{\alpha}_* \bar{\psi}_1 \bar{\psi}_2 \tag{3.9}$$

near critical points z_*. So our integral becomes

$$\begin{aligned} I &= \frac{1}{2\pi} \int e^{-(z,\psi_i)} d^2 z d^4 \psi = \sum_{z_*} \frac{1}{2\pi} \int e^{-|\alpha(z-z_*)|^2} |\alpha_*|^2 \psi_1 \psi_2 \bar{\psi}_1 \bar{\psi}_2 d^2 z d^4 \psi \\ &= \sum_{z_*} \frac{|\alpha_*|^2}{|\alpha_*|^2} = \sum_{z_*} 1, \end{aligned}$$

counting (not with sign) the number of critical points $\{z_*\}$ of S. More generally, let $f(z)$ be any holomorphic function. Then the (unnormalized) expectation value of $f(z)$ is

$$\langle f(z) \rangle = \int e^{-S(z,\psi_i)} f(z) d^2 z d^4 \psi.$$

But this expression is still invariant under $\bar{\delta}$ transformations, so it again localizes to the critical points of $\bar{W}(\bar{z})$. The expectation value of f therefore reduces to

$$\begin{aligned} \langle f(z) \rangle &= \sum_{z_*} f(z_*) \frac{1}{2\pi} \int e^{-S^{(2)}(z,\psi_1)} d^2 z d^4 \psi \\ &= \sum_{z_*} f(z_*). \end{aligned}$$

This relies crucially on the fact that $\bar{f} = 0$. Now since $\bar{Q}_i^2 = 0$, one way to construct any $\bar{Q}_i$-invariant function is to take $\bar{Q}_i$ of something, e.g. $\bar{Q}_i \Lambda(z, \bar{z}, \psi_j, \bar{\psi}_k)$ for some general Λ.

However, if $F = \bar{Q}\Lambda$, then the expectation value of F is

$$\langle F \rangle = \langle \bar{Q}\Lambda \rangle = \int (\bar{Q}\Lambda) e^{-S} \frac{d^2 z d^4 \psi}{2\pi} = \int \bar{Q} \left(\Lambda e^{-S} \right) \frac{d^2 z d^4 \psi}{2\pi} = 0, \tag{3.10}$$

where we have again moved e^{-S} into the $\bar{Q}$ (since it is $\bar{Q}$-invariant) and observed as before that since $\bar{Q} \sim \bar{\psi}\frac{\partial}{\partial z} + \partial W \frac{\partial}{\partial \bar{\psi}}$, the second term vanishes by the Berezin integration rules and the first term is a total derivative w.r.t z, and therefore is a vanishing boundary term.

Therefore interesting functions are in $H+_{\bar{Q}} = \frac{\ker \bar{Q}}{\mathrm{im} \bar{Q}}$, where $\ker\bar{Q}$ is the set of functions F with $\bar{Q}F = 0$ and $\mathrm{im}\bar{Q}$ is the set of functions $F = \bar{Q}\Lambda$ for any function Λ. Thus we can always decompose a general function into

$$\left\langle F + \bar{Q}\Lambda \right\rangle = \langle F \rangle + \left\langle \bar{Q}\Lambda \right\rangle = \langle F \rangle . \tag{3.11}$$

Suppose $F_i = \bar{Q}\Lambda$. Then

$$\left\langle \prod_{i=1}^{n} F_i \right\rangle = \left\langle \bar{Q}\Lambda \prod_{i=1}^{n} F_i \right\rangle = \left\langle \bar{Q}(\Lambda \left(\prod_{i=1}^{n} F_i \right)) \right\rangle = 0. \tag{3.12}$$

Lecture 4.

Thursday, January 31, 2019

Last time, we considered both the image and kernel of the operator $\bar{Q}$. We remarked that some functions will have non-trivial correlators from $O \in H_{\bar{Q}}$, the $\bar{Q}$ cohomology. That is, we are interested in functions that are in the kernel of $\bar{Q}$ ($\bar{Q}$-closed) but not in its image ($\bar{Q}$-exact).

For example, the transform $\bar{\delta}\bar{\psi}_i = \bar{\epsilon}_i \partial W$ shows that ∂W is itself $\bar{Q}$ of something, i.e. in the image. Thus if our operators O_i contain ∂W as a factor, their correlator vanishes, e.g. if

$$W(z) = \frac{z^{n+1}}{n+1} - az, \partial W = z^n - a \tag{4.1}$$

, then we have non-trivial $\bar{Q}$-invariant operators that are polynomials subject to the condition that $z^n = a$. This tells us that these operators form a ring generated by the set of functions $\{1, z, z^2, \ldots, z^{n-1}\}$. The ring of non-trivial SUSY operators is often called the *chiral ring* (chiral because we've made a choice of $\bar{Q}$ or Q).

Supersymmetric quantum mechanics There are (at least) two perspectives on QM: the canonical framework (with operators, states, wavefunctions) and the path integral framework. Today we will stay in the canonical framework and see what SUSY can teach us about quantum mechanics.

Take a worldline theory of a single bosonic field $x(t)$ and a single complex fermion $\psi(t)$ (plus its conjugate $\bar{\psi}$). We choose the action

$$S[x, \psi, \bar{\psi}] = \int \left[\frac{1}{2}\dot{x}^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{1}{2}(\partial h)^2 - \bar{\psi}\psi\partial^2 h \right] dt, \tag{4.2}$$

with $h = h(x(t))$ some potential function along the worldline as before. Now that we have one dimension, we have some kinetic terms in our action.

Now, this action $S[x, \psi, \bar{\psi}]$ is invariant under SUSY transformations

$$\delta x = \epsilon\bar{\psi} - \bar{\epsilon}\psi \tag{4.3}$$
$$\delta\psi = \epsilon(i\dot{x} + \partial h) \tag{4.4}$$
$$\delta\bar{\psi} = \bar{\epsilon}(-i\dot{x} + \partial h). \tag{4.5}$$

We'll defer a discussion of where these transformations actually come from to when we talk about superfields. For now, we'll just take it for granted that we can write down such transformations, and note that we should check explicitly the action is indeed invariant under this set of variations.

By the Noether procedure, promoting $\epsilon \to \epsilon(t)$, we find that

$$\delta S = -i \int (\dot{\epsilon} Q + \dot{\bar{\epsilon}}\bar{Q}) dt, \tag{4.6}$$

where the charges

$$Q = \bar{\psi}(i\dot{x} + \partial h), \quad \bar{Q} = \psi(-i\dot{x} + \partial h) \tag{4.7}$$

obey the following algebra:

$$\begin{aligned}\{Q,\bar{Q}\}x &= (Q\bar{Q}+\bar{Q}Q)x \\ &= -Q\psi+\bar{Q}\bar{\psi} \\ &= -(i\dot{x}+\partial h)+(-i\dot{x}+\partial h) \\ &= -2i\dot{x},\end{aligned}$$

and

$$\begin{aligned}\{Q,\bar{Q}\}\psi &= \bar{Q}(i\dot{x}+\partial h) && (4.8)\\ &= -i\dot{\psi}-\psi\partial^2 h && (4.9)\\ &\simeq -2i\dot{\psi} && (4.10)\end{aligned}$$

after applying the equation of motion $\dot{\psi}=-i\psi\partial^2 h$. Similarly,

$$\{Q,\bar{Q}\}=\bar{\psi}\simeq -2i\dot{\bar{\psi}}. \tag{4.11}$$

Thus up to the fermionic equations of motion, the anticommutator of the supercharges generates time translations and so must be $\propto H$ the Hamiltonian.

To canonically quantize, we have

$$p=\frac{\delta L}{\delta\dot{x}}=\dot{x},\quad \pi=\frac{\delta L}{\delta\dot{\psi}}=i\bar{\psi}. \tag{4.12}$$

Making the appropriate substitutions, we have a Hamiltonian

$$H=p\dot{x}+\pi\dot{\psi}-L=\frac{1}{2}p^2+(\partial h)^2+\frac{1}{2}\partial^2 h(\bar{\psi}\psi-\psi\bar{\psi}). \tag{4.13}$$

Note that classically, we could have just anticommuted $\bar{\psi}$ and ψ to get rid of the factor of $1/2$, but after quantization we will have to be more careful about ordering ambiguities. Upon quantization (in units where $\hbar=1$), we impose canonical commutation relations,

$$[x,p]=i,\quad \{\psi,\bar{\psi}\}=1. \tag{4.14}$$

For x, as usual we shall take it to lie in the Hilbert space $\mathcal{H}=L^2(\mathbb{R},dx)$, the space of square-integrable functions of a real variable, in which case

$$\hat{x}\Psi(x)=x\Psi(x)$$

and

$$\hat{p}\Psi(x)=-i\frac{\partial\Psi}{\partial x}.$$

The relations $\{\hat{\psi},\hat{\bar{\psi}}\}=1$ are now reminiscent of $[a,a^\dagger]=1$, the relation for the raising and lowering operators of the harmonic oscillator. In analogy to the harmonic oscillator, let's therefore define a fermionic number operator

$$\hat{F}=\hat{\bar{\psi}}\hat{\psi}. \tag{4.15}$$

Since $\hat{F}$ is the product of two fermionic operators, it is a bosonic operator, and we can then compute immediately that

$$[\hat{F},\hat{\psi}]=-\hat{\psi},\quad [\hat{F},\hat{\bar{\psi}}]=+\hat{\bar{\psi}}. \tag{4.16}$$

We also let the vacuum of the fermionic system be the state $|0\rangle$, defined by

$$\hat{\psi}|0\rangle=0. \tag{4.17}$$

The first excited state is naturally

$$\hat{\bar{\psi}}|0\rangle=|1\rangle.$$

However, since $\{\hat{\bar{\psi}},\hat{\bar{\psi}}\}=0$ (i.e. $\hat{\bar{\psi}}^2=0$), there are no further excited states. Hence the Hilbert space of the fermionic states is limited to these two states (up to a phase, of course) and the entire Hilbert space of the system is therefore

$$\mathcal{H}=L^2(\mathbb{R},dx)|0\rangle\oplus L^2(\mathbb{R},dx)|1\rangle. \tag{4.18}$$

We can equivalently write this as the sum

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F,$$

a bosonic part and a fermionic part. In the quantum theory, our SUSY operators become

$$\hat{Q} = \hat{\bar{\psi}}(i\hat{p} + \partial h), \quad \hat{\bar{Q}} = \hat{\psi}(-i\hat{p} + \partial h). \tag{4.19}$$

The quantum Hamiltonian is

$$\hat{H} = \frac{1}{2}\hat{p}^2 + (\partial h)^2 + \frac{1}{2}\partial^2 h(\hat{\bar{\psi}}\hat{\psi} - \hat{\psi}\hat{\bar{\psi}}). \tag{4.20}$$

Dropping hats (so that everything is an operator by assumption), we have immediately

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0. \tag{4.21}$$

However, the anticommutator of Q and $\bar{Q}$ is nontrivial. In fact,

$$\{Q, \bar{Q}\} = 2H, \tag{4.22}$$

which we leave as an exercise. This is why we made the choice of the particular ordering in defining H. This is the quantum analogue of the statement that the anticommutator of Q and $\bar{Q}$ gave us time translation in our classical variables. In the quantum theory, they yield the Hamiltonian.

Supersymmetric ground states As before, $\langle\Psi|H|\Psi\rangle \geq 0$, with equality iff $Q|\Psi\rangle = 0$ and $\bar{Q}|\Psi\rangle = 0$. Therefore a state of zero energy in super-quantum mechanics (SQM) must be SUSY invariant and will then be a ground state.

If we represent the fermionic vacuum $|0\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the excited state $|1\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then we can write the conditions on the operators $Q|\Psi\rangle = 0, \bar{Q}|\Psi\rangle = 0$ in a matrix representation,

$$\begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + \partial h & 0 \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0, \quad \begin{pmatrix} 0 & -\frac{d}{dx} + \partial h \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0. \tag{4.23}$$

Solving, we learn that our ground state must be of the form

$$|\Psi\rangle = \begin{pmatrix} Ae^{-h(x)} \\ Be^{+h(x)} \end{pmatrix}. \tag{4.24}$$

We want a normalizable solution, so we must set at least one of A, B to zero. This depends on the asyptotic behavior of $h(x)$:

- if $h(x) \to_{|x|\to\infty} +\infty$ then the SUSY ground state is $\begin{pmatrix} Ae^{-h(x)} \\ 0 \end{pmatrix}$.
- if $h(x) \to_{|x|\to\infty} -\infty$ then the SUSY ground state is $\begin{pmatrix} 0 \\ Be^{+h(x)} \end{pmatrix}$.
- if $h(x) \to_{x\to+\infty} \pm\infty$ and $h(x) \to_{x\to-\infty} \mp\infty$, then neither solution will be square-integrable, so there is no zero energy state. The ground state will have nonzero energy and SUSY is *spontaneously broken*.

Lecture 5.

Tuesday, February 5, 2019

Recall that we have non-negative energy states in SUSY since

$$\langle\Psi|H|\Psi\rangle = ||Q|\Psi\rangle||^2 + ||\bar{Q}|\Psi\rangle||^2 \geq 0.$$

Thus we argued that the supersymmetric ground states must be annihilated by both Q and $\bar{Q}$, and are either $e^{-h(x)}|0\rangle$ or $e^{+h(x)}|1\rangle$ if $h(x)$ is a polynomial of even degree, and no SUSY ground state exists if $h(x)$ is of odd degree.

Meanwhile, excited states in our theory, $E > 0$, come in pairs. If $\mathcal{H} = \bigoplus \mathcal{H}_n$, where $H|\Psi_n\rangle = E_n|\Psi_n\rangle\, \forall |\Psi_n\rangle \in \mathcal{H}_n$, then we can further split each of the Hilbert spaces $\mathcal{H}_n = \mathcal{H}_{B,n} \oplus \mathcal{H}_{F,n}$ into bosonic and fermionic parts. In particular $Q : \mathcal{H}_{n,F} \to \mathcal{H}_{n,B}$ (since $[Q, H] = 0$) and annihilates $\mathcal{H}_{n,B}$. Thus given $|b\rangle \in \mathcal{H}_{n,B}$ a bosonic state at energy level n, we have

$$2E_n|b\rangle = (Q\bar{Q} + \bar{Q}Q)|b\rangle = Q(\bar{Q}|b\rangle). \tag{5.1}$$

For $E_n > 0$, the RHS of this equation cannot be zero, so

$$|b\rangle = \frac{1}{2E_n} Q\bar{Q}|b\rangle = Q|f\rangle \tag{5.2}$$

where

$$|f\rangle \equiv \frac{\bar{Q}|b\rangle}{2E_n} \in \mathcal{H}_{n,F}. \tag{5.3}$$

That is, a bosonic state is Q of something (namely, a fermionic state). Similarly, any state in $\mathcal{H}_{n,F}$ with $n > 0$ can be written as $\bar{Q}|g\rangle$ for some $|g\rangle \in \mathcal{H}_{n,B}$. Thus

$$\mathcal{H}_{n,B} \cong \mathcal{H}_{n,F} \text{ when } n > 0 \tag{5.4}$$

and each excited state comes in pairs, with a bosonic and fermionic partner.

Definition 5.5. We define the *Witten index* to be the difference between the number of fermionic and bosonic ground states,

$$I_W = \dim \mathcal{H}_{0,B} - \dim \mathcal{H}_{0,F} = \mathrm{Tr}_{\mathcal{H}}(-1)^F = \mathrm{Tr}_{\mathcal{H}}\Big((-1)^F e^{-\beta H}\Big), \tag{5.6}$$

where the last two expressions follow because excited states come in pairs and F is the eigenvalue of the fermionic number operator.

Note the final expression is independent of β. One may ask why we add on this $e^{-\beta H}$ factor if $(-1)^F$ already counts the Witten index properly. One reason is to regularize the trace– while it is true that the excited states do come in pairs, the trace may be a bit ill-defined if the terms we are adding do not go to zero. Another reason is to make a connection to the path integral.

Path integrals in QM Consider a particle traveling on $\mathbb{R}$. The time evolution operator e^{-iHt} becomes $e^{-H\tau}$ under a Wick rotation (i.e. imaginary time) $t_{\text{Mink}} \to i\tau$. If our particle is at y_0 at $\tau = 0$, the amplitude to find it at y_1 at some later time $\tau = \beta$ is

$$\langle y_1|e^{-\beta H}|y_0\rangle = K_\beta(y_1, y_0) = \frac{1}{\sqrt{2\pi\beta}} \exp\left(-\frac{(y_0 - y_1)^2}{2\beta}\right). \tag{5.7}$$

This is sometimes known as the *heat kernel*. If we break this evolution in to steps of length $\Delta\tau = \beta/N$, we can rewrite the heat kernel as an integral over complete sets of states,

$$\langle y_1|e^{-\beta H}|y_0\rangle = \int \langle y_1|e^{-\Delta\tau H}|x_{N-1}\rangle\langle x_{N-1}|e^{-\Delta\tau H}|x_{N-2}\rangle \ldots \langle x_2|e^{-\Delta\tau H}|x_1\rangle\langle x_1|e^{-\Delta\tau H}|y_0\rangle d^{N-1}x. \tag{5.8}$$

However, this is none other than a set of heat kernels:

$$\begin{aligned}\langle y_1|e^{-\beta H}|y_0\rangle &= \int K_{\Delta\tau}(y_1, x_{N-1}) \ldots K_{\Delta\tau}(x_2, x_1) K_{\Delta\tau}(x_1, y_0) d^{N-1}x \\ &= \frac{1}{\sqrt{2\pi\Delta\tau}} \int \exp\left[-\sum_{i=0}^{n} \frac{\Delta\tau}{2}\left(\frac{x_{i+1} - x_i}{\Delta\tau}\right)^2\right] \prod_{i=1}^{N-1} \frac{dx_i}{\sqrt{2\pi\Delta\tau}}.\end{aligned}$$

Taking the limit $\Delta\tau \to 0, N \to \infty$ with fixed β, we define

$$\exp\left(-\int_0^\beta \frac{1}{2}\dot{x}^2 d\tau\right)\mathcal{D}x \equiv \lim_{\Delta\tau\to 0, N\to\infty} \prod_i \frac{dx_i}{\sqrt{2\pi\Delta\tau}} \exp\left[-\frac{\Delta\tau}{2}\sum_i\left(\frac{x - i + 1 - x_i}{\Delta\tau}\right)^2\right], \tag{5.9}$$

where we (heuristically) obtain the path integral representation

$$\langle y_1|e^{-\beta H}|y_0\rangle = \int_{\mathcal{C}[y_1, y_0]} e^{-\int_0^\beta \frac{1}{2}\dot{x}^2 d\tau}\mathcal{D}x, \tag{5.10}$$

where $\mathcal{C}[y_1, y_0]$ is the space of continuous maps $x : [0, \beta] \to \mathbb{R}$ s.t. $x(0) = y_0, x(\beta) = y_1$. We can also show that this derivation works for a Hamiltonian with a potential, $H = \frac{p^2}{2} + V(x)$ in which case the action becomes $S = \int \left[\frac{1}{2}\dot{x}^2 + V(x)\right] d\tau$.

Now, the partition function $Z(\beta)$ is closely related to the heat kernel:

$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}) = \int_{\mathbb{R}} \langle y|e^{-\beta H}|y\rangle dy \tag{5.11}$$

$$= \int \left[\int \mathcal{C}[y,y] e^{-S[x]} \mathcal{D}x \right] dy \tag{5.12}$$

$$= \int_{\mathcal{C}_{S^1}} e^{-S[x]} \mathcal{D}x, \tag{5.13}$$

where we consider continuous maps $x : S^1 \to \mathbb{R}$ since the start and endpoints are the same y.

Path integrals for fermions We have fermionic coherent states defined analogous to the harmonic oscillator coherent states, as

$$|\eta\rangle = e^{\hat{\bar{\psi}}\eta}|0\rangle, \tag{5.14}$$

where η is just a number and the chief property of such a state is that it is an eigenstate of the lowering operator, $\hat{\psi}|\eta\rangle = \eta|\eta\rangle$. These obey

$$1_{\mathcal{H}} = \int e^{-\bar{\eta}\eta} |\bar{\eta}\rangle\langle\eta| d^2\eta \tag{5.15}$$

and

$$\mathrm{Tr}(\hat{A}) = \int \langle -\bar{\eta}|\hat{A}|\eta\rangle e^{-\bar{\eta}\eta} d^2\eta, \tag{5.16}$$

such that the supertrace (i.e. a modified trace which accounts for fermionic and bosonic parts of the Hilbert space) obeys

$$\mathrm{STr}(A) = \mathrm{Tr}_{\mathcal{H}}((-1)^F A) = \int \langle \bar{\eta}|\hat{A}|\eta\rangle e^{-\bar{\eta}\eta} d^2\eta. \tag{5.17}$$

Using these and following the same procedure as for bosons, we can define a heat kernel on fermions. For eigenstates $|\chi\rangle, |\bar{\chi}'\rangle$, we have

$$\langle \bar{\chi}'|e^{-\beta H}|\chi\rangle = \int \bar{\chi}' e^{-\Delta\tau H}|\eta_{N-1}\rangle\langle\bar{\eta}_{N-1}|e^{-\Delta\tau H}|\eta_{N-2}\rangle \ldots \langle\bar{\eta}_n|e^{-\Delta\tau H}|\eta_1\rangle\langle\bar{\eta}_1|e^{-\Delta\tau H}|\chi\rangle \prod_{k=1}^{N-1} e^{-\bar{\eta}_k\eta_k} d^2\eta_k.$$

Let's now order the Hamiltonian (using commutators if necessary) so that all $\hat{\psi}$s appear to the right of all $\hat{\bar{\psi}}$s (sort of like normal ordering). Take one of these heat kernel factors. In the limit as $\Delta\tau \to 0$, we only need the first-order term in the exponential,

$$\begin{aligned}
\langle \bar{\eta}_{k+1}|e^{-\Delta\tau H(\hat{\bar{\psi}},\hat{\psi})}|\eta_k\rangle &= \langle\bar{\eta}_{k+1}|1 - \Delta\tau H(\hat{\bar{\psi}},\hat{\psi})|\eta_k\rangle \\
&= \langle\bar{\eta}_{k+1}|1 - \Delta\tau H(\bar{\eta}_{k+1},\eta_k)|\eta_k\rangle \\
&= e^{-\Delta\tau H(\bar{\eta}_{k+1},\eta_k)}\langle\bar{\eta}_{k+1}|\eta_k\rangle \\
&= e^{-\Delta\tau H(\bar{\eta}_{k+1},\eta_k)} e^{+\bar{\eta}_{k+1}\eta_k}.
\end{aligned}$$

Using this, we can evaluate our fermionic heat kernel. It is

$$\langle \bar{\chi}'|e^{-\beta H}|\chi\rangle = \lim_{N\to\infty,\Delta\tau\to 0} \int \exp\left(\sum_{k=1}^{N} \bar{\eta}_k\eta_{k-1} - \Delta\tau H(\bar{\eta}_k,\eta_{k-1}) \right) \prod_{k=1}^{N-1} e^{-\bar{\eta}_k\eta_k} d^2\eta_k \tag{5.18}$$

$$= \lim_{N\to\infty,\Delta\tau\to 0} \int \exp\left(-\sum_{k=1}^{N} \left[\bar{\eta}_k \frac{(\eta_k - \eta_{k-1})}{\Delta\tau} - H(\bar{\eta}_k,\eta_{k-1}) \right] \Delta\tau \right) e^{\bar{\eta}_N\eta_N} \prod_{k=1}^{N-1} d^2\eta_k, \tag{5.19}$$

where this extra factor $e^{\bar{\eta}_N\eta_N}$ has come from us rewriting the exponent in to look more like a discretized derivative of η.

Therefore

$$\langle \bar{\chi}'|e^{-\beta H}|\chi\rangle = \int e^{-S[\bar{\eta},\eta]} e^{\bar{\eta}(\beta)\eta(\beta)} \mathcal{D}\eta\mathcal{D}\bar{\eta} \tag{5.20}$$

where $\eta(0) = \chi, \eta(\beta) = \bar{\chi}'$ and $S[\bar{\eta},\eta]$ is the action $\int_0^\beta \bar{\eta}\dot{\eta} - H(\bar{\eta},\eta)$. When we compute the partition function, we find that

$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}) = \int \langle -\bar{\chi}|e^{-\beta H}|\chi\rangle e^{-\bar{\chi}\chi} d^2\chi = \exp(-S[\bar{\psi},\psi])\mathcal{D}\psi\mathcal{D}\bar{\psi} \tag{5.21}$$

where we now have *antiperiodic* boundary conditions, $\psi(\tau+\beta) = -\psi(\tau)$. Equivalently to the bosonic case we have a supertrace

$$\begin{aligned}\mathrm{STr}(e^{-\beta H}) = \mathrm{Tr}((-1)^F e^{-\beta H}) &= \int \langle \bar\chi | e^{-\beta H} | \chi \rangle e^{-\bar\chi \chi} d^2\chi \\ &= \int e^{-S[\bar\psi,\psi]} \mathcal{D}\psi \mathcal{D}\bar\psi\end{aligned}$$

with periodic boundary conditions.

Lecture 6.

Thursday, February 7, 2019

Last time, we defined the Witten index as the difference between the number of fermionic and bosonic ground states, and we wrote it in terms of a "supertrace" over (Boltzmann-like) factors $\mathrm{STr}(e^{-\beta H})$. We found that it admitted a path integral expression,

$$I_W = \int_{\text{periodic}} e^{-S_E[x,\psi,\bar\psi]} \mathcal{D}x \mathcal{D}\psi \mathcal{D}\bar\psi, \tag{6.1}$$

where S_E is the Euclidean action

$$S_E = \oint \left[\frac{1}{2}\dot{x}^2 + \bar\psi\dot\psi + \frac{1}{2}(\partial h)^2 + \partial^2 h \bar\psi\psi \right] d\tau \tag{6.2}$$

where dots now indicate $d/d\tau$ (i.e. with respect to Euclidean time). Note this action is invariant under the SUSY transformations

$$\delta x = \epsilon\bar\psi - \bar\epsilon\psi \tag{6.3}$$

$$\delta\psi = \epsilon(-\dot{x} + \partial h) \tag{6.4}$$

$$\delta\bar\psi = \bar\epsilon(\dot{x} + \partial h). \tag{6.5}$$

Note that these transformations only make sense globally on S^1 since $(x, \psi, \bar\psi)$ are all periodic and $\epsilon, \bar\epsilon$ are all constants. If we try to make this a local transformation, allowing $\epsilon(\tau + 2\pi) = -\epsilon(\tau)$ requires that we *gauge* these transforms, which leads to *supergravity*.

Let's now compute the Witten index I_W using the path integral. As in $d = 0$, we shall consider rescaling $h \to \lambda h$ for $\lambda \in \mathbb{R}_+$, and we expect that I_W is actually independent of this rescaling. Let's see this explicitly:

$$\frac{d}{d\lambda} I_W(\lambda) = -\int_P \left[\oint_{S^1} \lambda(\partial h)^2 + \partial^2 h \bar\psi\psi \right] e^{-S_E[x,\bar\psi,\psi]} \mathcal{D}x \mathcal{D}\psi \mathcal{D}\bar\psi.$$

However, note that

$$\begin{aligned} Q_\lambda(\oint \partial h \psi \, d\tau) &= \oint \left[\partial^2 h \bar\psi\psi + \lambda(\partial h)^2 - \partial h \frac{dx}{d\tau} \right] d\tau \\ &= \oint_{S^1} \lambda(\partial h)^2 + \bar\psi\psi\partial^2 h \, d\tau - \oint_{S^1} dh. \end{aligned}$$

But this last term is zero since it is a total derivative integrated around a closed loop. Therefore this insertion is Q_λ-exact, and we conclude that

$$\frac{dI_W(\lambda)}{d\lambda} = 0, \tag{6.6}$$

as expected from the canonical calculation. In particular, as $\lambda \to \infty$ the term $\exp\left(-\frac{\lambda^2}{2} \oint (\partial h)^2 d\tau \right)$ suppresses all maps $x : S^1 \to \mathbb{R}$ except in a neighborhood of constant maps to critical points of h.

Near such critical points, we may expand $x(\tau) = x_* + \delta x(\tau)$ so that to quadratic order,

$$S_E^{(2)} = \oint \frac{1}{2} \delta x \left(-\frac{d^2}{d\tau^2} + h''(x_*)^2 \right) \delta x + \bar\psi \left(\frac{d}{d\tau} + h''(x_*) \right) \psi d\tau. \tag{6.7}$$

Since $\delta x(\tau)$ and the fermions $\psi, \bar{\psi}$ must each be periodic, we can expand them as Fourier series,

$$\delta x(\tau) \sum_{n\in\mathbb{Z}} \delta x_n \exp\left(\frac{2\pi i n\tau}{\beta}\right), \quad \psi(\tau) = \sum_{n\in\mathbb{Z}} \psi_n \exp\left(\frac{2\pi i n\tau}{\beta}\right) \tag{6.8}$$

where the ψ_n are Grassmann quantities, as they must be, and $\delta x_{-n} = (\delta x_n)^*$ since $\delta x(\tau) \in \mathbb{R}$. We now find near a critical point x_* that we can explicitly perform the path integral:

$$\int e^{-S_E^{(2)}} \mathcal{D}\delta x \mathcal{D}\psi \mathcal{D}\bar{\psi} = \frac{\det(\partial_\tau + h''(x_*))}{\sqrt{\det(-\partial_\tau^2 + h''(x_*)^2}} \tag{6.9}$$

$$= \frac{\prod_{n\in\mathbb{Z}}(2\pi i n/\beta + h''(x_*))}{\sqrt{\prod_{n\in\mathbb{Z}}\left((2\pi n/\beta)^2 + h''(x_*)^2\right)}}. \tag{6.10}$$

But because we observed that the Fourier modes are paired up by $\delta x_{-n} = (\delta x_n)^*$, only the $n = 0$ terms will not cancel. We find that a single critical point therefore has

$$\int e^{-S_E^{(2)}} \mathcal{D}\delta x \mathcal{D}\psi \mathcal{D}\bar{\psi} = \frac{h''(x_*)}{|h''(x_*)|} \tag{6.11}$$

or summing over critical points,

$$I_W = \sum_{x_*:\partial h(x_*)=0} \frac{h''(x_*)}{|h''(x_*)|}. \tag{6.12}$$

This agrees precisely with our notion that the Witten index counts a topological property of h, namely the net number of critical points (counting $h'' > 0$ as 1 and $h'' < 0$ as -1).

Non-linear sigma models In the bosonic case, we let our field describe a map $x : M \to N$ from our worldline $M([0,\beta], S^1)$ to a compact Riemannian manifold (N, g). Often we let x^a be coordinates on some subset $U \subset N$, and $x^a(\tau)$ be the corresponding fields where $a = 1, \ldots, n = \dim(N)$.

We choose the following action

$$S[x] = \int_M \frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b \, d\tau. \tag{6.13}$$

Note that this metric $g_{ab}(x)$ generically depends on $x(\tau)$, so this is an interacting (worldline) QFT. That is, the zeroth order behavior would be a simple kinetic term, but we expect nonlinear corrections. Varying this action $S[x]$, we get

$$\delta S = \int_M \left[g_{ab}(x) \dot{x}^a \frac{d\delta x^b}{d\tau} + \frac{1}{2} \partial_c g_{ab} \dot{x}^a \dot{x}^b \delta x^c \right] d\tau \tag{6.14}$$

$$= \int \left[-\frac{d}{d\tau}(g_{ac}\dot{x}^a + \frac{1}{2}\partial_c g_{ab} \dot{x}^a \dot{x}^b) \right] \delta x^c d\tau + g_{ab}(x) \dot{x}^a \delta x^b |_{\partial M}. \tag{6.15}$$

However, notice that the equations of motion are the geodesic equations

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0, \tag{6.16}$$

where Γ is the Levi-Civita connection on (N, g). This is a nice classical result. Can we make it quantum?

To quantize, notice that

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab} \dot{x}^b, \tag{6.17}$$

so we get canonical commutation relations

$$[\hat{x}^a, \hat{p}_b] = i\delta^a{}_b.$$

We can moreover choose the Hilbert space to be $\mathcal{H} = L^2(N, \sqrt{g} d^n x)$, square integrable functions under the standard Riemannian volume element $\sqrt{g} d^n x$ on the manifold N.

We appear to have constructed a theory of a free particle moving on a curved manifold. However, there's no preferred choice of Hamiltonian when we quantize. Classically, we have (as usual)

$$H = p_a \dot{x}^a - L = \frac{1}{2} g^{ab}(x) p_a p_b, \tag{6.18}$$

but there's an ordering ambiguity when we turn this into a quantum operator because our metric depends on x.

We can start to address this by reasonably requiring the following:

- $\hat{H}$ should be generally covariant.
- $\hat{H}$ should reduce to $-\frac{1}{2}\frac{p^2}{\partial x^2}$ in the case $(N, g) = (\mathbb{R}^n, d^n x)$.
- $\hat{H}$ should contain no more than two derivatives acting either on the wavefunction $\Psi \in H$ or g.

In fact, there's a 1-parameter family of such $\hat{H}$s given by

$$\hat{H} = -\frac{1}{2}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^a}\left(g^{ab}\sqrt{g}\frac{\partial}{\partial x^b}\right) + \alpha R[g]\right) \tag{6.19}$$

for $\alpha \in \mathbb{R}$, where $R[g]$ is the Ricci scalar corresponding to the metric on our target space.

Beyond this, there is no preferred choice of α, and different regularizations of the path integral will give different values of α. To do better, we need to supersymmetrize this worldline model, and we'll do this next week.

Lecture 7.

Tuesday, February 12, 2019

A quick bit of admin– office hours are rescheduled to tomorrow (February 13) from 2-4 PM.

Last time, we wrote down a 1-parameter family of Hamiltonians

$$\hat{H} = -\frac{1}{2}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^a}\left(g^{ab}\sqrt{g}\frac{\partial}{\partial x^b}\right) + \alpha R[g]\right),$$

where this first term is none other than $-\frac{1}{2}\nabla^a\nabla_a$, a covariant Laplacian. However, we also get this extra term– we get a bit of coupling to the Ricci scalar which we're free to choose.

Supersymmetric NLSM As always, our goal will be to supersymmetrize the model and see what we learn. Take a bosonic variable x and fermionic variable ψ^a such that

$$x : M \to N, \quad \psi^a \in \prod \Omega^0(M, x^* T_N) \tag{7.1}$$

where N is some target Riemannian manifold (N, g), M is something like an interval $[0, \beta]$ or S^1, and Ω^0 is a function on the worldline.

With these variables, we can write down an action

$$S[x, \psi] = \int_M \left[\frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b + i g_{ab}\bar{\psi}^a(\nabla_t \psi)^b - \frac{1}{2}R_{abcd}\psi^a\bar{\psi}^b\psi^c\bar{\psi}^c\right] dt \tag{7.2}$$

where $\nabla_t \psi^a = \frac{d\psi^a}{dt} + \Gamma^a_{bc}\frac{dx^b}{dt}\psi^c$ is the pullback of the connection on N. This first term is just our bosonic kinetic term, and the second is a reasonable sort of fermionic kinetic term, but the last is a bit of a surprise. The fermions couple directly to the Riemann curvature of the target space N.

Note that this action is invariant under the SUSY transformations

$$\delta x^a = \epsilon\bar{\psi}^a - \bar{\epsilon}\psi^a \tag{7.3}$$

$$\delta\psi^a = \epsilon(i\dot{x}^a - \Gamma^a_{bc}\bar{\psi}^b\psi^c) \tag{7.4}$$

$$\delta\bar{\psi}^a = \bar{\epsilon}(-i\dot{x}^a - \Gamma^a_{bc}\bar{\psi}^b\psi^c), \tag{7.5}$$

which we can check explicitly (e.g. on the second examples sheet). As before, the origin of these transformations may seem a bit mysterious but will become clearer when we discuss superfields.

These transformations are generated by the Noether charges $Q, \bar{Q}$, where

$$Q = i\bar{\psi}^a(g_{ab}\dot{x}^b + ig_{bc}\bar{\psi}^b\Gamma^c_{ad}\psi^d), \tag{7.6}$$
$$\bar{Q} = -i\psi^a(g_{ab}\dot{x}^b + ig_{bc}\psi^b\Gamma^c_{ad}\bar{\psi}^d). \tag{7.7}$$

Our action is also invariant under the (considerably simpler) transformation $\psi^a \mapsto e^{i\alpha}\psi^a, \bar{\psi}^a \mapsto e^{-i\alpha}\bar{\psi}^a$, generated by the charge $F = g_{ab}\psi^a\bar{\psi}^b$. Conservation of F in the quantum theory tells us that no fermionic excitations are created or destroyed by time evolution.

Quantizing this theory, we have the conjugate momenta

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab}\dot{x}^b + ig_{bc}\bar{\psi}^b\Gamma^c_{ad}\psi^d; \quad \pi_a = \frac{\delta L}{\delta \dot{\psi}^a} = ig_{ab}\bar{\psi}^b. \tag{7.8}$$

These have canonical (anti)commutation relations

$$[\hat{x}^a, \hat{p}_b] = i\delta^a{}_b, \quad \{\hat{\psi}^a, \hat{\bar{\psi}}^b\} = g^{ab}(x), \tag{7.9}$$

with all others trivial. For the bosonic fields, we choose the Hilbert space

$$\mathcal{H} = L^2(N, \sqrt{g}d^nx),$$

with $\hat{p}_a \to -i\frac{\partial}{\partial x^a}$. For the fermions, we again choose $\bar{\psi}^a$ to be raising operators and ψ^a lowering operators. If we then pick a vacuum $|0\rangle$ defined by $\psi^a|0\rangle = 0 \forall a$, all other states of the fermionic system are generated by acting with the $\bar{\psi}$s on $|0\rangle$.

Notice that each $\bar{\psi}^a$ can only act once, since $\{\bar{\psi}^a, \bar{\psi}^b\} = 0$. This is because our whole universe is just an interval (remember, $M = [0, \beta]$ or S^1), so rather than having Fourier modes defined on a time slice, we have different fields defined at points. This enforces the Pauli exclusion principle, if you like. However, we do get different fermions since we have an index $\bar{\psi}^a$ to range over. We can interpret these as *forms* on N: thus

$$|0\rangle \leftrightarrow 1 \tag{7.10}$$
$$\bar{\psi}^a|0\rangle \leftrightarrow dx^a \tag{7.11}$$
$$\bar{\psi}^a\bar{\psi}^b|0\rangle \leftrightarrow dx^a \wedge dx^b \tag{7.12}$$
$$\bar{\psi}^a \dots \bar{\psi}^n|0\rangle \leftrightarrow dx^1 \wedge dx^2 \wedge \dots \wedge dx^n. \tag{7.13}$$

There are no more since any $(\bar{\psi}^a)^2 = 0$.

Altogether, the Hilbert space of SUSY QM is thenrefore

$$\mathcal{H} = \Omega^{\cdot}(N) = \bigoplus_{p=0}^{n} \Omega^p(N) \tag{7.14}$$

where $\Omega^p(N)$ is the space of p-forms on N, i.e. a general state $\Psi(x, \bar{\psi})$ can be written as

$$\Psi(x, \bar{\psi}) = f(x) + \alpha_a(x)\bar{\psi}^a + \beta_{ab}(x)\bar{\psi}^a\bar{\psi}^b + \dots + \omega_{1\dots n}(x)\bar{\psi}^1 \dots \bar{\psi}^n \tag{7.15}$$
$$\sim f(x) + \alpha_a dx^a + \beta_{ab} dx^a \wedge dx^b + \dots + \omega dx^a \wedge \dots \wedge dx^n. \tag{7.16}$$

So there is a direct correspondence between expanding in $\bar{\psi}$ operators and in terms of differential forms.

Acting on this space, we can set up the full correspondence

$$\hat{x}^a \to x^a \times (\cdot)$$
$$\hat{p}_a \to -i\frac{\partial}{\partial x^a}(\cdot)$$
$$\bar{\psi}^a \to dx^a \wedge (\cdot)$$
$$\psi^a \to g^{ab}\frac{\partial}{\partial x^b}(\cdot) \text{ (contraction)}.$$

Thus when we hit some arbitrary state $\bar{\psi}^a\bar{\psi}^b\bar{\psi}^c\ldots\bar{\psi}^d|0\rangle$ with a lowering operator (where there are an odd number of $\bar{\psi}$s), we can use the anticommutation relations to find

$$\begin{aligned}\psi^e(\bar{\psi}^a\bar{\psi}^b\bar{\psi}^c\ldots\bar{\psi}^d|0\rangle) &= \{\psi^e,\bar{\psi}^a\bar{\psi}^b\ldots\bar{\psi}^d\}|0\rangle\\ &= \Big(\{\psi^e,\bar{\psi}^a\}\bar{\psi}^b\ldots\bar{\psi}^d - \bar{\psi}^a\{\psi^e,\bar{\psi}^b\}\bar{\psi}^c\ldots\bar{\psi}^d+\ldots+\bar{\psi}^a\bar{\psi}^b\ldots\{\psi^e,\bar{\psi}^d\}\Big)|0\rangle\\ &= \Big(g^{ea}\bar{\psi}^b\ldots\bar{\psi}^d - g^{eb}\bar{\psi}^a\bar{\psi}^c\ldots\bar{\psi}^d+\ldots+g^{ed}\bar{\psi}^a\bar{\psi}^b\ldots\Big)|0\rangle.\end{aligned}$$

But notice this is just what we would have gotten from contracting

$$\iota_{g^{ef}\frac{\partial}{\partial x^f}}(dx^a\wedge dx^b\ldots\wedge dx^d). \tag{7.17}$$

The inner product on $\mathcal{H}$ is

$$\langle\alpha|\beta\rangle = \int_N \alpha\wedge *\beta, \tag{7.18}$$

where $*$ is the Hodge star operator.[2] Here, if $\alpha,\beta\in\Omega^p(N)$ (p-forms on N), then

$$\int_N \alpha\wedge*\beta = \int_N \alpha^{a_1\ldots a_p}\beta_{a_1\ldots a_p}\sqrt{g}, \tag{7.19}$$

with indices raised using g^{ab}. For $\alpha\in\Omega^p(N),\beta\notin\Omega^p(N)$, we simply define $\int_N\alpha\wedge*\beta=0$. This follows since ψ^a is the adjoint of $\bar{\psi}^a$, so

$$\int_N \bar{\alpha}_{a_1\ldots a_p}(x)\beta_{b_1\ldots b_p}(x)\sqrt{g}d^nx\ \underbrace{\langle 0|\psi^{a_1}\ldots\psi^{a_p}}_{\text{fermionic part of }\langle\alpha|}\ \underbrace{\bar{\psi}^{b_1}\ldots\bar{\psi}^{b_p}|0\rangle}_{\text{fermionic part of }|\beta\rangle} = \int_N \bar{\alpha}^{a_1\ldots a_p}\beta_{a_1\ldots a_p}\sqrt{g}d^nx. \tag{7.20}$$

We see that the integral vanishes unless we precisely match the indices between α and β.

Furthermore, in the quantum theory we get

$$Q = i\bar{\psi}^a\hat{p}_a \to dx^a\frac{\partial}{\partial x^a} = d, \tag{7.21}$$

where d is now the exterior derivative, taking us from $\Omega^p(N)\to\Omega^{p+1}(N)$, from p-forms to $p+1$-forms. Similarly,

$$\bar{Q} = -i\psi^a\hat{p}_a \to d^\dagger, \tag{7.22}$$

where $d^\dagger$ is the adjoint with respect to the inner product $\langle,\rangle$ which takes us from $\Omega^p(N)\to\Omega^{p-1}(N)$, from p to $p-1$-forms, such that if $\alpha\in\Omega^p,\beta\in\Omega^{p+1}$, then $\langle\alpha,d^\dagger\beta\rangle = \langle d\alpha,\beta\rangle$.

Lecture 8.

Thursday, February 14, 2019

We've been looking at supersymmetric nonlinear sigma models. Previously, our fields were maps from $x: M\to N$ where M was a worldline and N was some target space, a Riemannian manifold with a metric g. But it's clear that M could be some bigger manifold, in general "our universe."

We said the Hilbert space for our theory was

$$\mathcal{H} = \mathcal{H}_x\otimes\mathcal{H}_\psi = \Omega^{\cdot}(N,\mathbb{C}), \tag{8.1}$$

the space of differential forms up to p-forms on N equipped with inner product

$$\langle\alpha|\beta\rangle = \int_N \bar{\alpha}\wedge*\beta, \tag{8.2}$$

where $*$ is the Hodge star operator taking $\Omega^p(N)\to\Omega^{n-p}(N)$. Explicitly, if

$$\omega = \omega_{a_1a_2\ldots a_p}dx^{a_1}\wedge dx^{a_2}\wedge\ldots\wedge dx^{a_p},$$

[2] No, we didn't learn enough of this differential geometry in *General Relativity*. Blame Malcolm Perry. I direct you to Harvey Reall's GR notes and/or Sean Carroll's excellent textbook *Spacetime and Geometry* as a reference.

then $*\omega$ is given by

$$*\omega = \frac{\sqrt{g}}{(n-p)!}\epsilon^{a_1\ldots a_p}{}_{b_{p+1}\ldots b_n}\omega_{a_1\ldots a_p}dx^{b_{p+1}}\wedge\ldots\wedge dx^{b_n},$$

with indices raised by the inverse metric. We saw that our SUSY operator Q then has the geometric interpretation of an exterior derivative,

$$\hat{Q} = i\hat{\bar{\psi}}^a\hat{p}_a \leftrightarrow d, \tag{8.3}$$

and similarly $\hat{\bar{Q}}$ has the interpretation of the adjoint of the exterior derivative,

$$\hat{\bar{Q}} = -i\hat{\psi}^a\hat{p}_a \leftrightarrow d^\dagger, \tag{8.4}$$

where $\langle\alpha, d^\dagger\beta\rangle = \langle d\alpha, \beta\rangle$.

We can now fix the ordering ambiguity in $|hatH$ by demanding the SUSY algebra

$$2\hat{H} = \{\hat{Q}, \hat{\bar{Q}}\} \tag{8.5}$$

still holds in the quantum theory. This fixes

$$H = \frac{1}{2}(d^\dagger d + dd^\dagger) = -\frac{1}{2}\Delta, \tag{8.6}$$

where Δ is the Laplacian acting on forms. Since $d:\Omega^p\to\Omega^{p+1}, d^\dagger:\Omega^p\to\Omega^{p-1}$, it follows that $-\Delta = d^\dagger d + dd^\dagger : \Omega^p \to \Omega^p$.

To see this concretely, when acting on a function $f\in\Omega^0(N)$ (i.e. a zero-form), $d^\dagger$ simply annihilates the function (since there are no -1-forms) so we get

$$\begin{aligned}
-\Delta f &= d^\dagger df\\
&= d^\dagger(\partial_a f dx^a)\\
&= *d(*df)\\
&= *d\left(\frac{\sqrt{g}}{(n-1)!}g^{ab}\partial_a f\epsilon_{bc\ldots d}\underbrace{dx^c\wedge\ldots\wedge dx^d}_{n-1}\right)\\
&= \frac{*}{(n-1)!}\partial_m(\sqrt{g}g^{ab}\partial_a f)\epsilon_{bc\ldots d}\underbrace{dx^m\wedge dx^c\wedge\ldots\wedge dx^d}_{n}.
\end{aligned}$$

But we see that there are now n one-forms being wedged together, which means we must have all the dx^1 through dx^n in some order. We can rewrite this as a totally antisymmetric tensor, with a factor of $1/g$ the determinant of the metric. Using this fact, our expression becomes

$$\begin{aligned}
-\Delta f &= \frac{1}{g}\partial_b(g^{ab}\sqrt{g}\partial_a f)*(dx^1\wedge\ldots\wedge dx^n)\\
&= -\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b f).
\end{aligned}$$

What we learn is that the generalized Laplacian acting on forms reduces to the ordinary Laplacian with respect to the metric when acting on functions.

However, we now observe that acting on any form ω,

$$\begin{aligned}
2\langle\omega|\hat{H}|\omega\rangle &= \langle\omega|dd^\dagger\omega\rangle + \langle\omega|d^\dagger d\omega\rangle\\
&= ||d^\dagger\omega||^2 + ||d\omega||^2 \geq 0.
\end{aligned}$$

A form which has equality here, $\Delta\omega = 0$, is said to be *harmonic*. Therefore supersymmetric ground states are in 1 : 1 correspondence with $\text{Harm}^\cdot(N) = \oplus_{p=0}^n \text{Harm}^p(N)$, the space of harmonic 0- through p-forms on N. Notice that any form $\omega\in\text{Harm}^p$ must be closed ($d\omega = 0$) and co-closed ($d^\dagger\omega = 0$).

Theorem 8.7 (Hodge's theorem)**.** *The space of harmonic p-forms on N is in correspondence with the de Rham p-cohomology group,*

$$\text{Harm}^p(N) \cong H^p_{dR}(N) \tag{8.8}$$

where

$$H^p_{dR}(N) = \{\omega \in \Omega^p(N) \text{ s.t. } d\omega = 0\}/\{\omega = d\alpha\} = \ker(d : \Omega^p \to \Omega^{p+1})/\mathrm{im}(d : \Omega^{p-1} \to \Omega^p). \tag{8.9}$$

In de Rham cohomology, ω is specified up to $\omega \sim \omega + d\alpha$ (i.e. we only care about ω up to the addition of some exact $d\alpha$). The role of the co-closure condition, $d^\dagger \omega = 0$, is to select a unique representative. If $d\omega = d^\dagger\omega = 0$, then we our freedom becomes $\omega \sim \omega + d\alpha$ where $d^\dagger d\alpha = 0$, and the only solutions are $\alpha = 0$.[3] Thus the space of SUSY ground states is $\cong H^{\cdot}_{dR}(N)$.

Thinking back to our discussion of the Witten index, we see that

$$I_W = \mathrm{Tr}((-1)^F e^{-\beta H}) = n_B - n_F = \sum_{p=0}^{n} (-1)^p \dim(H^p_{dR}(N)). \tag{8.10}$$

But this is very interesting because this final expression is precisely $\chi(N)$, the *Euler character* of N. Thus the space of SUSY ground states has a close relation to some topological information about the space our states live in.

To motivate de Rham cohomology a bit more, suppose C_p is a p-cycle in N without boundary. Stokes's Theorem in the vector calculus language says that

$$\int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}.$$

But we can generalize this to p-forms:

$$\int_{D_{p+1}} d\omega = \int_{C_p} \omega \tag{8.11}$$

if $\partial D_{p+1} = C_p$. That is, we can relate the integral in some region D_{p+1} to the value of the form integrated over the boundary C_p. However, if $\omega \in H^p_{dR}$, then $d\omega = 0 \implies \int \omega = 0$ if C_p is the boundary of some D_{p+1}.

Furthermore,

$$\int_{C_p} \omega + d\alpha = \int_{D_{p+1}} d\omega + \int_{D_{p+1}} d^2\alpha, \tag{8.12}$$

where this second term vanishes since d is nilpotent. Thus we arrive at de Rham's theorem:

Theorem 8.13 (de Rham)**.**

$$H^p_{dR}(N) \cong H_p(N), \tag{8.14}$$

where $H_p(N)$ denotes the pth homology group, the set {p-cycles in N with no boundary}/{p-cycles that are the boundary of some $(p+1)$-cycle}.

For instance, if $N = S^n$, then $\dim(H^0_{dR}(S^n)) = 1$. We can also find that $\dim(H^p_{dR}(S^n)) = 0$ for $p \neq 0, n$ since we can contract any loop (e.g. an S^1) to a point on S^n. And then we have $\dim(H^n_{dR}(S^n)) = 1$, i.e. there is one non-trivial "wrapping" of S^n by an S^n.

For $n = \Sigma_g$ a handlebody with genus n (i.e. n donuts glued together) we have instead $H^0(\Sigma_g) = \mathbb{C}$, $H^1(\Sigma_g) = \mathbb{C}^{2g}$ and $H^2(\Sigma_g) = \mathbb{C}$ (dimensions $1, 2g$, and 1).

The Euler character for the n-sphere is

$$\chi(S^n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd}, \end{cases} \tag{8.15}$$

and for Σ_g it is $\chi(\Sigma_g) = 2 - 2g$.

For the path integral, $\chi(N) = \int e^{-S[x,\psi]} \mathcal{D}x \mathcal{D}\psi \mathcal{D}\bar\psi$, where all fields are periodic with period β. Now if we take the whole action, we see that our whole action is supersymmetrically trivial:

$$S = \int \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b + \frac{1}{2} g_{ab} \bar\psi^a \nabla_t \psi^b + \frac{1}{4} R_{abcd} \bar\psi^a \psi^b \bar\psi^c \psi^d dt \tag{8.16}$$

$$= \bar{Q}\left[\oint \frac{g_{ab}\bar\psi^a}{2} (i\dot{x}^b + \Gamma^b_{cd} \bar\psi^c \psi^d) dt.\right] \tag{8.17}$$

[3]This is equivalent to the procedure in electromagnetism where we have a potential with a gauge symmetry $A \sim A + d\lambda$, and we fix the gauge by requiring that $d^\dagger A = \partial^\mu A_\mu = 0$.

We therefore learn that the path integral is independent of β.

Lecture 9.

Tuesday, February 19, 2019

Recall that we wrote down an action last time in the path integral, which took the form

$$S = \int \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b + \frac{1}{2} g_{ab}\bar{\psi}^a \nabla_t \psi^b + \frac{1}{4} R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d dt,$$

and we saw that the Witten index I_W can be written in terms of the de Rham cohomology groups as well as the Euler characteristic:

$$I_W = \sum_{p=0}^{\dim N} (-1)^p \dim(H^p_{dR}(N)) = \chi(N).$$

Since our action is $\bar{Q}$-exact, the path integral is independent of the circumference β of S^1. In particular, if we expand our fields as

$$x^a(\tau) = x^a_0 + \delta x^a(\tau) \text{ with } \oint \delta x^a(\tau) d\tau = 0 \tag{9.1}$$

$$\psi^a(\tau) = \psi^a_0 + \delta\psi^a(\tau) \text{ with } \oint \delta\psi^a(\tau) d\tau = 0 \tag{9.2}$$

then as $\beta \to 0$, all the contributions from the non-zero modes $(\delta x^a, \delta\psi^a)$ are highly suppressed, e.g.

$$\delta x^a(\tau) = \sum_{k\neq 0} \delta x^a_k e^{2\pi i k\tau/\beta} \tag{9.3}$$

where these $\delta x^a(\tau)$ have the interpretation of Fourier modes, and derivatives bring down $1/\beta \to \infty$. In fact, the contributions from $\delta x, \delta\psi$ precisely cancel each other, leaving us with just an integral over the zero-modes (x_0, ψ_0). That is, the path integral localizes as before to constant maps $x_0 : S^1 \to N$, but there's no preferred point in N in the absence of a potential, so we still need to integrate over N.

The Witten index is then given by a path integral over the zero modes

$$\begin{aligned}\chi(N) &= \int e^{-S[x_0,\psi_0]} d^n x_0 d^n \psi_0 d^n \bar{\psi}_0 \\ &= \int \exp - \left[\frac{1}{2} R_{abcd}(x_0)\bar{\psi}^a_0\psi^b_0\bar{\psi}^c_0\psi^d_0\right] d^n x_0 d^n\psi_0 d^n\bar{\psi}_0 \\ &= \int_N \mathrm{Tr}(\underbrace{R \wedge R \wedge \ldots \wedge R}_{n\text{-form part}})\end{aligned}$$

where $R^a{}_b = R_{cd}{}^a{}_b dx^c \wedge dx^d$ is the curvature 2-form. But looking at the form of the exponential, we notice that $\chi(N) = 0$ for any theory with an odd number of fermionic zero modes. Therefore $\chi(N) = 0$ if $\dim(N)$ is odd. This is the *Gauss-Bonnet formula* (up to a constant we've neglected).

Aatiyah-Singer index theorem We take $\dim(N) = n = 2m$ to be even-dimensional. We can then restrict the fermions ψ^a to be real, which allows us to simplify the action– it becomes

$$S[x,\psi] = \oint \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b + \frac{i}{2} g_{ab}\psi^a\nabla_\tau\psi^b d\tau, \tag{9.4}$$

where the last term has vanished since $R_{a[bcd]} = 0$ by the Bianchi identity.

This action is still invariant under SUSY transforms with $\epsilon = -\bar{\epsilon}$, i.e.

$$\delta x^a = \epsilon\psi^a, \quad \delta\psi^a = -\epsilon\dot{x}^a. \tag{9.5}$$

This is sometimes called $N = 1/2$ SUSY in $d = 1$. We have momenta

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab}\dot{x}^b + \frac{i}{2}\psi_c\Gamma^c_{ab}\psi^b \tag{9.6}$$

where $\psi_c = g_{cd}\psi^d$ and we've picked up the second term from the $\dot{x}$ hiding inside the covariant derivative. The other momentum is

$$\pi_a = \frac{\delta L}{\delta \dot{\psi}^a} = g_{ab}\psi^b. \tag{9.7}$$

Thus we have canonical commutation relations

$$[\hat{x}^a, \hat{p}_b] = i\delta^a{}_b, \quad \{\hat{\psi}^a, \hat{\psi}^b\} = 2g^{ab}. \tag{9.8}$$

Note that here we don't have relations between $\psi, \bar{\psi}$ since the ψs are now real. Thus we won't have some elements with the natural interpretation of raising and lowering operators; instead, we will get some objects which look like spinors in $n = 2m$ dimensions.

For our purposes, the Dirac γ matrices obey $(\gamma^i)^\dagger = \gamma^i$ and $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$, where $i, j = 1, \ldots, \dim(N)$ are tangent (flat) indices. We construct $m = n/2$ raising and lowering operators over $\mathbb{C}$ by taking

$$\gamma^I_\pm = \frac{1}{2}(\gamma^{2I} \pm i\gamma^{2I+1}) \quad \text{for } I = 1, \ldots, m \tag{9.9}$$

where we combine the even gamma matrices with the next odd ones (with a $\pm$ sign respectively). One may check that these obey

$$\{\gamma^I_+, \gamma^J_-\} = \delta^{IJ}, \quad \{\gamma^I_+, \gamma^J_+\} = 0, \quad \text{and } \{\gamma^I_-, \gamma^J_-\} = 0. \tag{9.10}$$

So these really are raising and lowering operators.

Starting from a spinor χ that obeys $\gamma^I_-\ chi = 0 \forall I$ (effectively a vacuum state), we construct a basis of the space S of spinors by acting with any combination of the raising operators γ^I_+. However, since $(\gamma^I_+)^2 = 0$, each γ^I_+ can act at most once, so $\dim(S) = 2^{n/2}$ (since each γ^I_+ either acts or does not act on this vacuum state).

The group $\mathrm{Spin}(n)$ (the double cover of $SO(n)$) then acts on these spinors via the generators

$$\Sigma^{ij} = -\frac{1}{4}[\gamma^i, \gamma^j], \tag{9.11}$$

which themselves obey

$$[\Sigma^{ij}, \Sigma^{kl}] = i\Big(\delta^{ik}\delta^{jl} - \delta^{il}\Sigma^{ik} - \delta^{jk}\Sigma^{il} - \delta^{il}\Sigma^{jk}\Big). \tag{9.12}$$

This representation is not irreducible. Let $\gamma^{n+1} = i^{n/2}\gamma^1\gamma^2 \ldots \gamma^n$ (equivalent to γ^5 in the usual case). This obeys

$$(\gamma^{n+1})^2 = 1, \quad \{\gamma^{n+1}, \gamma^i\} = 0, \quad \text{and } [\gamma^{n+1}, \Sigma^{ij}] = 0. \tag{9.13}$$

We therefore decompose the space of spinors S as

$$S = S^+ \oplus S^-$$

where $S^\pm$ are the ± 1 eigenspaces of γ^{n+1} and correspond to states constructed from an even/odd number of raising operators γ^I_+ acting on χ.

The Dirac operator $i\partial\!\!\!/$ anticommutes with γ^{n+1}, and thus decomposes as

$$i\partial\!\!\!/ = \begin{pmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{pmatrix} \tag{9.14}$$

where $\partial^\pm : S^\pm \to S^\mp$. Note that $\partial^\pm$ annhilates $S^\mp$, so $(\partial^\pm)^2 = 0$. This should remind us a bit of the exterior derivative. We now define

$$\mathrm{index}(i\partial\!\!\!/) = \dim\ker(\partial^+) - \dim\ker(\partial^-). \tag{9.15}$$

In our quantization of

$$S = \int \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b + \frac{i}{2}g_{ab}\psi^a\nabla_\tau\psi^b d\tau,$$

the Hilbert space is thus naturally $L^2(S(N), \sqrt{g}d^n x)$, and the supercharge

$$Q = \psi^a(ig_{ab}\dot{x}^b + \psi_c\Gamma^c_{ab}\psi^b) \tag{9.16}$$

corresponds to the covariant Dirac operator $i\nabla\!\!\!\!/$. The Witten index $\mathrm{Tr}((-1)^F e^{-\beta H})$ is then just the index of the Dirac operator split into its chiral parts,

$$\mathrm{Tr}((-1)^F e^{-\beta H}) = \dim\ker(\nabla^+) - \dim\ker(\nabla^-). \tag{9.17}$$

The path integral is again independent of the circumference β. Splitting

$$x^a(\tau) = x_0^a + \delta x^a(\tau) \text{ with } \oint \delta x^a(\tau) d\tau = 0$$
$$\psi^a(\tau) = \psi_0^a + \delta\psi^a(\tau) \text{ with } \oint \delta\psi^a(\tau) d\tau = 0$$

as before, we use Riemann normal coordinates near $x_0 \in N$ to write the metric as the flat metric (in Euclidean signature) up to an $O(\delta x^2)$ correction,

$$g_{ab}(x) = \delta_{ab} - \frac{1}{3} R_{acbd}(x_0)\delta x^c \delta x^d + O(\delta x^3). \tag{9.18}$$

The connection may be chosen to vanish to zeroth order and to be given in terms of the curvature to first order:

$$\Gamma^a_{bc}(x) = \partial_d \Gamma^a_{bc}(x_0)\delta x^d = -\frac{1}{3}(R^a{}_{bcd}(x_0) + R^a{}_{cbd}(x_0))\delta x^d + O(\delta x^2). \tag{9.19}$$

To second order in the fluctuations, the action becomes

$$S^{(2)}[x_0, \psi_0, \delta x, \delta\psi] = \oint \left(-\frac{1}{2}\delta x_a \frac{d^2}{d\tau^2}\delta x^a + \frac{1}{2}\delta\psi_a \frac{d}{d\tau}\delta\psi^a - \frac{1}{4} R_{abcd}\psi_0^a \psi_0^b \delta x^c \delta\dot{x}^d \right) d\tau. \tag{9.20}$$

Lecture 10.

Thursday, February 21, 2019

Last time, we considered fields in some spacetime and chose Gaussian normal coordinates in order to write (for variations of the fields $x^a = x_0^a + \delta x^a(\tau), \psi^a = \psi_0^a + \delta\psi^a(\tau)$,

$$g_{ab}(x) = \delta_{ab} - \frac{1}{3} R_{acbd}(x_0)\delta x^c \delta x^d + O(\delta x^3)$$

and a connection

$$\Gamma^a_{bc}(x) = \partial_d \Gamma^a_{bc}(x_0)\delta x^d = -\frac{1}{3}(R^a{}_{bcd}(x_0) + R^a{}_{cbd}(x_0))\delta x^d + O(\delta x^2).$$

So we have the quadratic action

$$S^{(2)}[x_0, \psi_0, \delta x, \delta\psi] = \oint \left(-\frac{1}{2}\delta x^a \delta_{ab} \frac{d^2}{d\tau^2}\delta x^b + \frac{1}{2}\delta\psi^a \delta_{ab} \frac{d}{d\tau}\delta\psi^b - \frac{1}{4} R_{abcd}\psi_0^a \psi_0^b \delta x^c \frac{d\delta x^d}{d\tau} \right) d\tau. \tag{10.1}$$

For any fixed (x_0^a, ψ_0^a), this is a free action, so the path integral over fluctuations gives

$$\int e^{-S[x_0,\psi_0,\delta x,\delta\psi]}\mathcal{D}\delta x \mathcal{D}\delta\psi = \frac{\sqrt{\det'(\partial_\tau \delta_a^b)}}{\sqrt{\det'(-\partial^2\tau\delta_b^a - \mathcal{R}^a{}_b(x_0,\psi_0)\partial_\tau)}} \tag{10.2}$$

where $\mathcal{R}^a{}_b = R^a{}_{bcd}(x_0)\psi_0^c\psi_0^d$ and $\det'$ means without zero modes, i.e. we haven't yet done the integrals over (x_0, ψ_0).

We can split up the denominator by pulling out a ∂_τ to find

$$\int e^{-S[x_0,\psi_0,\delta x,\delta\psi]}\mathcal{D}\delta x \mathcal{D}\delta\psi = \frac{\sqrt{\det'(\partial_\tau \delta_a^b)}}{\sqrt{\det'(\delta^a{}_b\partial_\tau)}\sqrt{\det'(-\delta^a{}_b\partial_\tau - \mathcal{R}_a{}^b)}} = \frac{1}{\sqrt{\det'(-\delta^a{}_b\partial_\tau - \mathcal{R}^a{}_b)}}. \tag{10.3}$$

Notice that the matrix $\mathcal{R}^a{}_b$ is an antisymmetric $n \times n$ matrix (since we contracted over two indices in the original Riemann tensor, and $R^a{}_{bcd}$ was already antisymmetric in the first two indices) and $n = 2m$. We therefore decompose the tangent space $TN|_{x_0}$ into m 2-dimensional subspaces on which $\mathcal{R}^a{}_b|_i$ takes the form

$$\mathcal{R}^a{}_b|_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}. \tag{10.4}$$

Let $-D_i$ be the restriction of $-\delta^a{}_b\partial_\tau - \mathcal{R}^a{}_b$ to this 2D subspace.

We expand

$$\delta x^a(\tau) = \sum_{k\neq 0} \delta x^a_k e^{2\pi i k\tau}. \tag{10.5}$$

Then the eigenvalues of $-D_i$ on this subspace are $-2\pi i k \pm \omega_i$ for $k \in \mathbb{Z}, k \neq 0$ (where the first term comes from acting on a Fourier mode with ∂_τ and the second comes the eigenvalues of $\mathcal{R}^a{}_b|_i$ being $\pm\omega$). Therefore

$$\begin{aligned}\det(-D_i) &= \prod_{k\neq 0}(-2\pi i k + \omega_i)(-2\pi i k - \omega_i)\\ &= \prod_{k\neq 0}(-(2\pi k)^2 - \omega_i^2)\\ &= \prod_{k=1}^{\infty}(2\pi k)^4 \prod_{k=1}^{\infty}\left(1 + \frac{\omega_i^2}{(2\pi k)^2}\right)^2,\end{aligned}$$

where the rewriting in the last line has come from changing the $k \neq 0$ product to a product over $k = 1 \to \infty$.

This is clearly divergent thanks to the first factor. However, we can regularize this, e.g. using zeta-function regularization. We find that

$$\prod_{k=1}^{\infty}(2\pi k)^4 = (4\pi^2)^{2\zeta(0)} e^{-2\zeta'(0)} = 1. \tag{10.6}$$

The important factor is then

$$\prod_{k=1}^{\infty}\left(1 + \frac{\omega_i^2}{(2\pi k)^2}\right)^2,$$

and we recall that

$$\sinh(z) = z\prod_{k=1}^{\infty}\left(1 + \frac{z^2}{\pi^2 k^2}\right),$$

so after regularization, we have that $z = \omega_i^2/2$ and (by direct comparison with the expansion of $\sinh(z)$) our determinant term can be written as

$$\sqrt{\det{}'(-D_i)} = \frac{\sinh(\omega_i/2)}{(\omega_i/2)}. \tag{10.7}$$

We now see that

$$I_W = \mathrm{index}(\not{\nabla}) = \int \prod_{i=1}^{\infty} \frac{\omega_i/2}{\sinh(\omega_i/2)} d^n x_0 d^n \psi_0 \tag{10.8}$$

$$= \int \det\left(\frac{\mathcal{R}^a{}_b(x_0,\psi_0)/2}{\sinh(\mathcal{R}^a{}_b}(x_0,\psi_0)/2)\right) d^n x_0 d^n \psi_0. \tag{10.9}$$

where $\not{\nabla}$ denotes the Dirac operator on N. But by our regular Grassmann tricks, we must have precisely n factors of ψ_0 in order for this integral to be non-vanishing. Thus

$$I_W = \int_N \det\left(\frac{\mathscr{R}/2}{\sinh \mathscr{R}/2}\right). \tag{10.10}$$

where $\mathscr{R}^a{}_b = R^a{}_{bcd}(x) dx^c \wedge dx^d$ is a curvature two-form. This is the Aatiyah-Singer index theorem.

Supersymmetric QFT If we had a d-dimensional theory that is Lorentz invariant, we must complete the supersymmetry algebra $\{Q, Q^\dagger\} = 2H$. The Hamiltonian now comes with nontrivial kinetic terms and is part of the d-momentum multiplet P_μ, so we need further supercharges. If we want to preserve $Q^\dagger = (Q)^\dagger$, then these supercharges must have the same spin, and so must each have spin $1/2$.

Specifically, the SUSY algebra in d-dimensions is

$$\{Q_\alpha, Q^\dagger_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu, \tag{10.11}$$

where α, β are spinor indices and γ^μ is a Dirac γ matrix. We'll mostly be concerned with $d = 2$, where Dirac spinors have $2^{(d/2)} = 2$ complex components. Thus we can write $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$. With coordinates $(t, s) \in \mathbb{R}^2$ and Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(+, -)$, we can represent the Dirac γs as

$$\gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{10.12}$$

These obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The action for a free, massless Dirac spinor in $d = 2$ is then

$$S[\psi] = \frac{1}{2\pi} \int_{\mathbb{R}^2} i\bar{\psi}\partial\!\!\!/\psi d^2x \tag{10.13}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ and $\bar{\psi} = \psi^\dagger \gamma^t$. We can of course plug in the explicit form of the spinors and γ matrices, and we find that

$$S[\psi] = \frac{1}{2\pi} \int \mathbb{R}^2 i\bar{\psi}_-(\partial_t + \partial_s)\psi_- + i\bar{\psi}_+(\partial_t - \partial_s)\psi_+ dtds, \tag{10.14}$$

so we see that the spinor components decouple. Classically,

$$(\partial_t + \partial_s)\psi_- = 0 \implies \psi_-(t, s) = f(t - s) \tag{10.15}$$

represents a right-moving mode, while

$$(\partial_t - \partial_s)\psi_+ = 0 \implies \psi_+(t, s) = f(t + s) \tag{10.16}$$

is a left-moving mode. Under an $SO(1, 1)$ transformation, i.e.

$$\begin{pmatrix} t \\ s \end{pmatrix} \mapsto \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} \tag{10.17}$$

with γ the usual (real) rapidity, the spinor components transform as

$$\psi_\pm \mapsto e^{\pm\gamma/2}\psi, \quad \bar{\psi}_\pm \mapsto e^{\pm\gamma/2}\bar{\psi}. \tag{10.18}$$

Lecture 11.

Tuesday, February 26, 2019

Superspace in $d = 2$ Let $\mathbb{R}^{2/4}$ denote the superspace with coordinates $(x^0, x^1; \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$. Under an $SO(1, 1)$ transformation, the bosonic coordiantes transform as

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \mapsto \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}, \tag{11.1}$$

whereas the fermionic coordinates transform as spinors,

$$\theta^\pm \mapsto e^{\pm\gamma/2}\theta^\pm \tag{11.2}$$

$$\bar{\theta}^\pm \mapsto e^{\pm\gamma/2}\bar{\theta}^\pm. \tag{11.3}$$

We therefore introduce fermionic derviatives

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm \frac{\partial}{\partial x^\pm} \tag{11.4}$$

$$\bar{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm \frac{\partial}{\partial x^\pm} \tag{11.5}$$

where $\partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$,

These derivatives obey the anticommutation relations $\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} = -2i\partial_\pm$, so they represent our supersymmetry algebra on $\mathbb{R}^{2/4}$.

The SUSY transformations act geometrically on $\mathbb{R}^{2/4}$, being generaed by

$$\delta = \epsilon_+\mathcal{Q}_- - \epsilon_-\mathcal{Q}_+ - \bar{\epsilon}_+\bar{\mathcal{Q}}_- + \bar{\epsilon}_-\bar{\mathcal{Q}}_+, \tag{11.6}$$

where we note that the parameters $\epsilon_\pm, \bar{\epsilon}_\pm$ must themselves be spinors in order for $\Phi \to \Phi + \delta\Phi$.

Definition 11.7. A *superfield F* is simply a function on $\mathbb{R}^{2/4}$.

A generic superfield has an expansion

$$F(x\pm,\theta^{\pm},\bar{\theta}^{\pm}) = f_0(x^{\pm}) + \theta^+ f_+(x^{\pm}) + \theta^- f_-(x^{\pm}) + \bar{\theta}^+ g_+(x^{\pm}) + \bar{\theta}^- g_-(x^{\pm}) + \ldots + \theta^+\bar{\theta}^+\theta^-\bar{\theta}^- D(x^{\pm}). \tag{11.8}$$

This exapansion has $2^4 = 16$ components altogether (since each fermionic variable can either be there or not there).

Notice that under a SUSY transform $F \mapsto F + \delta F$, the highest component field $D(x^{\pm})$ can change at most by bosonic derivatives. We can see this by looking at the forms of the fermionic derivatives– since this is the coefficient of all the θs and $\bar{\theta}$s, there are no higher terms to bring down. The component for $\theta^+\bar{\theta}^+\theta^-$ could come up under $\mathcal{Q}_{\pm}$ but only after a $\frac{\partial}{\partial x^{\pm}}$. So indeed D is only changed up to bosonic derivatives.

Chiral superfields It's often useful to have smaller superfields that are constrained in some way. For this purpose, let us introduce

$$D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm} \tag{11.9}$$

$$\bar{D}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm}. \tag{11.10}$$

These are very similar to the $\mathcal{Q}$s, but they obey slightly different anticommutation relations:

$$\{D_{\pm}, \bar{D}_{\pm}\} = +2i\partial_{\pm}, \tag{11.11}$$

with other anticommutators zero. Moreover, it turns out that

$$\{D, Q\} = \{\bar{D}, Q\} = 0 \tag{11.12}$$

(for any choice of $\pm$ subscripts).

Definition 11.13. A *chiral superfield* Φ is a superfield which obeys $\bar{D}_{\pm}\Phi = 0$.

These Φ can depend on $(x^{\pm}, \theta^{\pm}, \bar{\theta})$ only through the combinations $(y^{\pm}, \theta^{\pm})$ where

$$y^{\pm} = x^{\pm} - i\theta^{\pm}\bar{\theta}^{\pm} \tag{11.14}$$

since $\bar{D}_{\pm}y^{\pm} = 0, \bar{D}_{\pm}y^{\mp} = 0$.

We can then expand a chiral superfield as

$$\Phi = \phi(y^{\pm}) + \theta^+\psi_+(y^{\pm}) + \theta^-\psi_-(y^{\pm}) + \theta^+\theta^- F(y^{\pm}). \tag{11.15}$$

Notice that the product $\Phi_1\Phi_2$ of any two superfields is again chiral, while the conjugate $\bar{\Phi}$ of a chiral superfield Φ obeys $D_{\pm}\bar{\Phi} = 0$ and is called *antichiral*.

Under a SUSY transformation $\Phi \mapsto \Phi + \delta\Phi$, but since all $\{Q, D\} = 0$, this SUSY transformation itself is chiral,

$$\bar{D}_{\pm}(\delta\Phi) = \delta(\bar{D}_{\pm}\Phi) = 0. \tag{11.16}$$

Thus SUSY transforms preserve chirality in this sense. To work out the SUSY transformations on the component fields, first note that

$$\mathcal{Q}_{\pm} = \left.\frac{\partial}{\partial \theta^{\pm}}\right|_{x,\bar{\theta}} + i\bar{\theta}^{\pm} \left.\frac{\partial}{\partial x^{\pm}}\right|_{\theta,\bar{\theta}} = \frac{\partial}{\partial \theta^{\pm}}|_{y,\bar{\theta}} + \frac{\partial y^{\pm}}{\partial \theta^{\pm}}|_{x,\bar{\theta}} \frac{\partial}{\partial y^{\pm}}|_{\theta,\bar{\theta}} + i\bar{\theta}^{\pm}\frac{\partial}{\partial y^{\pm}}|_{\theta,\bar{\theta}}. \tag{11.17}$$

But we see that since $\frac{\partial y^{\pm}}{\partial \theta^{\pm}}|_{x,\bar{\theta}} = -i\bar{\theta}^{\pm}$, these last two terms cancel and so

$$\mathcal{Q}_{\pm} = \frac{\partial}{\partial \theta^{\pm}}|_{y,\bar{\theta}}. \tag{11.18}$$

Similarly,

$$\bar{\mathcal{Q}}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}}|_{y,\theta} - 2i\theta^{\pm}\frac{\partial}{\partial y^{\pm}}|_{\theta,\bar{\theta}}. \tag{11.19}$$

Using this, one finds the component transformations

$$\delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+ \tag{11.20}$$

$$\delta\psi_{\pm} = \epsilon_{\pm}F \pm \bar{\epsilon}_{\mp}\partial_{\pm}\phi \tag{11.21}$$

$$\delta F = -2i\bar{\epsilon}\partial_-\psi_+ - 2i\bar{\epsilon}_-\partial_+\psi_-. \tag{11.22}$$

Thus the bosons are picking up some fermionic contributions, while the fermions pick up derivatives of the bosonic fields. There's also a bit of this mysterious F function which is influenced by derivatives of the fermions. Note that the SUSY transform of the θ^2 term F is a bosonic total derivative, in direct analogy to the D component of the superfield in 11.8.

Supersymmetric invariant actions The fact that the D-term of a generic superfield and F term of a chiral superfield vary only by total derivatives allows us to readily construct SUSY-invariant actions. Let $K(F_i, \Phi^a, \bar{\Phi}^a)$ be any real, smooth functions of real superfields $F_i(x^\pm, \theta^\pm, \bar{\theta}^\pm)$ and chiral superfields $\Phi^a(x^\pm - i\theta^\pm\bar{\theta}^\pm, \theta^\pm)$. Then

$$\int_{\mathbb{R}^{2/4}} K(F_i, \Phi^a, \bar{\Phi}^a)\, d^2x d^2\theta d^2\bar{\theta}$$

is SUSY invariant provided the component fields behave appropriately as $|x^\pm| \to \infty$. Integrating with respect to θs and $\bar{\theta}$s picks out the highest order term (the Fs and Ds), and we know that these transform at most up to a total derivative in the xs, so will be invariant after integrating with respect to d^2x. This K field is called the *Kähler potential*.

Likewise, suppose $W(\Phi^a)$ (called the *superpotential*) is a holomorphic function of Φ^a. Then

$$\bar{D}_\pm W(\Phi^a) = 0 \tag{11.23}$$

and so

$$\int_{\mathbb{R}^{2/4}} W(\Phi^a)\, d^2y d^2\theta \tag{11.24}$$

is again SUSY-invariant.

The Wess-Zumino model in $d = 1 + 1$ Let's consider the simplest case of a single chiral superfield Φ and its conjugate $\bar{\Phi}$. We take

$$K(\Phi, \Phi) = \bar{\Phi}\Phi \tag{11.25}$$

and keep the superpotential $W(\Phi)$ generic. Our action is then

$$S[\Phi, \bar{\Phi}] = \underbrace{\int_{\mathbb{R}^{2/4}} \bar{\Phi}\Phi\, d^2x d^4\theta}_{\text{kinetic terms}} + \left[\underbrace{\int_{\mathbb{R}^{2/2}} W(\Phi) d^2y d^2\theta + \text{c.c.}}_{\text{potential terms}} \right], \tag{11.26}$$

and this action is guaranteed to be supersymmetric (c.c. indicates complex conjugate of the superpotential), given appropriate asymptotics.

Lecture 12.

Thursday, February 28, 2019

Last time, we stated the Wess-Zumino model of a chiral superfield,

$$S[\Phi, \bar{\Phi}] = \int_{\mathbb{R}^{2/4}} \bar{\Phi}\Phi\, d^2x d^4\theta + \int_{\mathbb{R}^{2/2}} W(\Phi) d^2y d^2\theta + \int_{\mathbb{R}^{2/2}} \bar{W}(\bar{\Phi}) d^2\bar{y} d^2\bar{\theta}. \tag{12.1}$$

Note that $W(\Phi)$ is a holomorphic function of the superfield

$$\Phi = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm) \tag{12.2}$$

where $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$. Note also the limits of integration and the integration measures, since the superpotential separates into holomorphic and antiholomorphic parts. We have

$$W(\Phi)|_{\theta^+\theta^-} = F\partial W(\phi) - \psi_+\psi_- \frac{\partial^2 W}{\partial \phi^2}(\phi). \tag{12.3}$$

For the Kähler potential $|\Phi|^2_{\theta^4}$ we need to write

$$\Phi(x^\pm,\theta^\pm,\bar\theta^\pm) = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^-F(y^\pm) \tag{12.4}$$

$$= \phi(x^\pm) - i\theta^+\bar\theta^+\partial_+(x^\pm) - i\theta^-\bar\theta^-\partial_-\phi(x^\pm) + \theta^+\bar\theta^+\theta^-\bar\theta^-\partial_+\partial_-\phi(x^\pm) \tag{12.5}$$

$$+\theta^+\psi_+(x^\pm) - i\theta^+\theta^-\bar\theta^-\partial_-\psi_+(x^\pm) + \theta^-\psi_-(x^\pm) - i\theta^-\theta^+\bar\theta^+\partial_+\psi_-(x^\pm) + \theta^+\theta^-F(x^\pm) \tag{12.6}$$

as a function on non-chiral superspace. All we've done is expand $y^\pm$ in $x^\pm,\theta$, dropping any terms that are zero.

Similarly, the antiholomorphic part $\bar\Phi(x^\pm,\theta^\pm,\bar\theta^\pm$ has an expansion which looks like

$$\Phi(x^\pm,\theta^\pm,\bar\theta^\pm) = \bar\phi(x^\pm) + i\theta^+\bar\theta^+\partial_+(x^\pm) + i\theta^-\bar\theta^-\partial_-\bar\phi(x^\pm) - \theta^+\bar\theta^+\theta^-\bar\theta^-\partial_+\partial_-\bar\phi(x^\pm)$$
$$-\bar\theta^+\bar\psi_+(x^\pm) - i\bar\theta^+\theta^-\bar\theta^-\partial_-\bar\psi_+(x^\pm) - \bar\theta^-\bar\psi_-(x^\pm) - i\bar\theta^-\theta^+\bar\theta^+\partial_+\bar\psi_-(x^\pm) + \bar\theta^+\bar\theta^-\bar F(x^\pm).$$

We need to extract the $\theta^2\bar\theta^2$ term from $\bar\Phi\Phi$, i.e. we need to collect terms with all four θs.[4] We have

$$\begin{aligned}\bar\Phi\Phi|_{\theta^4} = &-\bar\phi\partial_+\partial_-\phi + \partial_+\bar\phi\partial_-\phi + \partial_-\bar\phi\partial_+\phi - \partial_+\partial_-\bar\phi\phi\\ &+ i\bar\psi_+\partial_-\psi_+ - i\partial_-\bar\psi_+\psi_+ + i\bar\psi_-\partial_+\psi_- - i\partial_+\bar\psi_-\psi_- + |F|^2.\end{aligned}$$

where we've been careful to reorder the θs to be in the order $\theta^+\theta^-\bar\theta^+\bar\theta^-$ to fix the signs.

Combining all the pieces we have a component action

$$\begin{aligned}S[\phi,\psi,F] = \int_{\mathbb{R}^2}&[\partial^\mu\bar\phi\partial_\mu\phi + i\bar\psi_-\partial_+\psi_- + i\bar\psi_+\partial_-\psi_+ + |F|^2\\ &+ FW'(\phi) - \psi_+\psi_-W''(\phi) + \bar F\bar W'(\bar\phi) - \bar\psi_-\bar\psi_+\bar W''(\bar\phi)]d^2x,\end{aligned} \tag{12.7}$$

where we've explicitly performed the integral over the θs. The kinetic terms come from $|\phi|^2_{\theta^4}$, while the potential terms come from the superpotential $W(\Phi),\bar W(\bar\Phi)$.

Notice the field F is auxiliary (i.e. its equation of motion is purely algebraic), so we can eliminate it using its equation of motion,

$$F + \bar W'(\bar\phi) = 0 \implies F = -\frac{\partial\bar W}{\partial\bar\phi}. \tag{12.8}$$

This gives us the interactions

$$\int -|W'(\phi)|^2d^2x = \int -V(\phi)d^2x, \tag{12.9}$$

giving a potential $V(\phi) = |W'(\phi)|^2$ for the scalars.

Symmetries of the WZ model By construction, this model is invariant under SUSY transformations acting on the component fields of Φ. The Noether currents for the supersymmetry are $G^\mu_\pm$ where

$$G^0_\pm = 2\partial_\pm\bar\phi\psi_\pm \pm i\bar\psi_\pm F \tag{12.10}$$

$$G^1_\pm = \mp 2\partial_\pm\bar\phi\psi_\pm + i\bar\psi_\mp F \tag{12.11}$$

and similarly for $\bar G^\mu_\pm$, the Noether charge is $Q_\pm = \int_{\mathbb{R}^1} G^0_\pm dx^1$, the integral over a constant time slice (remember we're in $1+1$ dimensions). Notice that the $G^\mu_\pm$ currents have spin $3/2$, so the charges $Q_\pm \mapsto e^{+\gamma/2}Q_\pm$ are each spin $1/2$, as expected for supercharges.

Consider the axial $U(1)_A$ transformation acting on $\Phi(x^\pm,\theta^\pm,\bar\theta^\pm)$ such that

$$\Phi(x^\pm,\theta^\pm,\bar\theta^\pm) \mapsto \Phi(x^\pm, e^{\mp i\alpha}\theta^\pm, e^{\pm i\alpha}\bar\theta^\pm), \tag{12.12}$$

leaving $\theta^+\theta^-$ invariant. Then $W(\Phi)|_{\theta^2}$ is likewise invariant, as is $\bar\Phi\Phi|_{\theta^4}$, so these transformations are also symmetries.

In terms of the component fields, we can equivalently think of these as

$$\phi \mapsto \phi, \quad \psi_\pm \mapsto e^{\mp i\alpha}\psi_\pm, \quad F \mapsto F. \tag{12.13}$$

[4] Gotta catch 'em all.

Writing it like this, it is clear looking at the form of the action that 12.7 is invariant under such transformations. The corresponding Noether charge is

$$F_A = \int_{\mathbb{R}^1} (\bar{\psi}_+\psi_+ - \bar{\psi}_-\psi_-)dx^1. \tag{12.14}$$

Now consider the $U(1)_V$ transformations

$$\Phi(x^\pm, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{iq\beta}\Phi(x^\pm, e^{-i\beta}\theta^\pm, e^{+i\beta}\bar{\theta}^\pm), \tag{12.15}$$

where θ^+, θ^- transform together, and we allow the whole superfield Φ to have charge q. In this case, $\theta^+\theta^-$ is not invariant but transforms to

$$\theta^+\theta^- \mapsto e^{-2i\beta}\theta^+\theta^-, \tag{12.16}$$

although the combination

$$\theta^2\bar{\theta}^2 \mapsto \theta^2\bar{\theta}^2, \tag{12.17}$$

is invariant. So the Kähler term is invariant for any q, whereas the superpotential term will only be invariant if

$$W \mapsto e^{2i\beta}W \tag{12.18}$$

to cancel the phase from the transformation of θ^2. In particular, for a monomial $W(\Phi) = c\Phi^k$, we have $U(1)_V$ symmetry iff we assign charge $q = 2/k$ to Φ. At the level of the component fields, these transformations can be taken to be

$$\phi \mapsto e^{2i\beta/k}\phi, \quad \psi_\pm \mapsto e^{(2/k-1)i\beta}\psi_\pm, \quad F \mapsto e^{(2/k-2)i\beta}F. \tag{12.19}$$

These are automatically symmetries of the kinetic terms (everything through $|F|^2$ in 12.7) since they are just phases, but we require this form to preserve the potential terms. What we'll show next time is that the superpotential is not altered by quantum corrections, so the quantum theory with respect to the superpotential has the same form as in the classical theory.

Lecture 13.

Tuesday, March 5, 2019

Previously, we considered the Wess-Zumino model and identified two $U(1)$ symmetries, the axial $U_A(1)$ symmetry sending

$$\phi \mapsto \phi, \quad \psi_\pm \mapsto e^{\mp i\alpha}\psi_\pm, \quad \bar{\psi}_\pm \mapsto e^{\pm i\alpha}\bar{\psi}_\pm \tag{13.1}$$

and the vector $U_V(1)$ symmetry sending

$$\phi \mapsto e^{iq\beta}\phi, \quad \psi_\pm \mapsto e^{i(q-1)\beta}\psi_\pm, \quad \bar{\psi}_\pm \mapsto e^{-i(q-1)\beta}\bar{\psi}_\pm. \tag{13.2}$$

Vacuum moduli space The scalar potential has the form

$$V(\phi) = |W'(\phi)|^2$$

or more generally

$$V(\phi^a, \bar{\phi}^{\bar{a}}) = \sum_a \left|\frac{\partial W}{\partial \phi^a}\right|^2. \tag{13.3}$$

Recall that a ground state $|\Omega\rangle$ is supersymmetric iff $H|\Omega\rangle = 0$. In particular, a field configuration can be a ground state if it sits in a (global) minimum of $V(\phi)$ over all space since $V \geq 0$. THus the ground state is supersymmetric if

$$\frac{\partial W}{\partial \phi^a}(\phi_0^a) = 0 \,\forall \phi^a,$$

i.e. if each of the terms in the potential sum 13.3 vanish.

In the quantum theory, the values ϕ_0^a are the expectation values $\phi_0^a = \langle\Omega|\phi^a|\Omega\rangle$ of the fields ϕ^a in the vacuum state. Sometimes these values are just zero, but we know that sometimes symmetries are broken and we expand about some nonzero field values. Typically, the holomorphic function $W(\phi^a)$ is a polynomial (over $\mathbb{C}$), so the vacuum conditions $\frac{\partial W}{\partial \phi^a}$ are a system of (complex) polynomials. The space of vacua $\mathcal{M}$ is the zero set of these polynomials, i.e.

$$\frac{\partial W}{\partial \phi^1} = \frac{\partial W}{\partial \phi^2} = \ldots = \frac{\partial W}{\partial \phi^n} = 0, \tag{13.4}$$

which defines a *affine algebraic variety*, so we make contact with algebraic geometry.

Example 13.5. Consider the superpotential

$$W(\phi) = m\frac{\phi^2}{2} + \lambda\frac{\phi^3}{3}. \tag{13.6}$$

The vacuum conditions are then

$$\frac{\partial W}{\partial \phi} = m\phi + \lambda\phi^2 = 0, \tag{13.7}$$

which tells us that either $\phi = 0$ or $\phi = -m/\lambda$. If we plot $V(\phi) = |W'|^2$, we see that there are two isolated supersymmetric minima in our theory.

Example 13.8. Consider now a theory with two fields l, h:

$$W(l, h) = \frac{\lambda}{2} l h^2. \tag{13.9}$$

Then the vacuum equations tell us

$$\frac{\partial W}{\partial l} = \frac{\lambda h^2}{2}, \quad \frac{\partial W}{\partial h} = \lambda l h. \tag{13.10}$$

Thus the vacuum moduli space requires $h = 0$ but l is unconstrained. However, for $\langle l \rangle \neq 0$ (the field l takes on a VEV), the field h is then massive with mass $|\lambda \langle l \rangle|$, whereas the field l is always massless in the vacuum (since $\langle h \rangle = 0$, and so there is no term which goes as $|l|^2$ in the action).

Example 13.11. Finally, consider a theory with superpotential

$$W(X, Y, Z) = XYZ. \tag{13.12}$$

In this case, we have conditions

$$\partial_X W = YZ, \quad \partial_Y W = XZ, \partial_Z W = XY. \tag{13.13}$$

We see that if any pair of the fields vanish, then all three vacuum conditions are satisfied, and the third field is free to take on any value we like. Therefore

$$\mathcal{M} = \{X = Y = 0\} \cup \{X = Z = 0\} \cup \{Y = Z = 0\}. \tag{13.14}$$

There are three "branches" of the space of vacua since e.g if we take $\langle X \rangle = \langle Y \rangle = 0$ then we are otherwise free to select $Z \in \mathbb{C}$.

In general, the vacuum moduli space is the affine variety $\mathbb{C}[\phi^1, \ldots, \phi^n]/(\partial_a W)$. If $\mathcal{M}$ is not just a set of isolated points, we say the potential $V(\phi^a)$ has *flat directions*, i.e. we can change some $\langle \phi^a \rangle$ continuously without leaving $V(\langle \phi^a \rangle) = 0$.

In a generic QFT, the structure of the classical potential is changed by quantum corrections. Couplings run with scale, and new couplings are generated (at least in an effective theory). In particular, these corrections tend to lift flat directions, leaving us with isolated vacua. SUSY theories are special and preserve the symmetry which gave rise to the original flat directions.

Seiberg Non-Renormalization Theorems In a supersymmetric theory, the effective superpotential $W_{\text{eff}}(\Phi)$ in the Wilsonian action (after integrating out modes) is actually identical to $W(\Phi)$. We can understand this with an example. Suppose $W(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$. Recall that our interactions come from $V(\phi) = |W'|^2$, so we will get some different vertices, and there are many non-trivial Feynman diagrams we can draw. (Note that fermionic couplings come from the other terms in the Wess-Zumino model.) For example, the 1-loop corrections to the m coupling receives contributions from some loop diagrams. However, these diagrams cancel exactly, and the same cancellation holds to all orders in λ! This is a remarkable simplification.

It appears as though $W(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$ breaks the $U_V(1)$ symmetry, and the $U_A(1)$ symmetry acts trivially on ϕ, so cannot help constrain the form of m^2_{eff}. So what saves our theory from loop corrections?

Seiberg's idea was to promote the couplings (m, λ) to chiral superfields (M, Λ) such that m, λ are the VEVs of the scalars in M, Λ. Note that (M, Λ) must be *chiral* superfields since they appear in $W(\Phi, M, \Lambda)$. In promoting these couplings to fields, we give them kinetic terms

$$K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + \frac{1}{\epsilon}[\bar{M}M + \bar{\Lambda}\Lambda], \tag{13.15}$$

so the new kinetic terms come with a factor $1/\epsilon$. Hence fluctuations in M, Λ are strongly suppressed as $\epsilon \to 0$.

The point of doing this is that $W(\Phi, M, \Lambda) = \frac{M}{2}\Phi^2 + \frac{\Lambda}{3}\Phi^3$ does preserve both $U_A(1), U_V(1)$ if we assign charges to the new superfields:

	Φ	M	Λ
$U_V(1)$	1	0	-1
$U_{WZ}(1)$	1	-2	-3

where $U_{WZ}(1)$ is an additional $U(1)$ symmetry acting trivially on $\theta^{\pm}, \bar{\theta}^{\pm}$. Provided we choose a regularization which is supersymmetric and preserves these two two $U(1)$s, the $W_{\text{eff}}(\Phi, M, \Lambda)$ in the Wilsonian action is constrained, so that it

- is holomorphic in (Φ, M, Λ)
- has $U_V(1)$ charge $+2$ and $U_{WZ}(1)$ charge zero
- reduces to the classical $W(\Phi)$ in the limit that $M, \Lambda \to 0$ (weak coupling).

Lecture 14.

Thursday, March 7, 2019

We continued our study of the Wess-Zumino model, with Seiberg's insight that we could promte the couplings m, λ to superfields in their own right, arriving at a superpotential

$$W(\Phi, M, \Lambda) = \frac{M}{2}\Phi^2 + \frac{\Lambda}{3}\Phi^3, \tag{14.1}$$

where $W_{\text{eff}}(\Phi, M, \Lambda)$ must be

- holomorphic in Φ, M, Λ
- $U_V(1)$ charge 2 and invariant under $\Phi \to e^{i\alpha}\Phi, M \to e^{-2i\alpha}M, \Lambda \to e^{-3i\alpha}\Lambda$
- reduce to $W(\Phi, \Lambda, M$ as $\Lambda \to 0$.

The first two conditions fix

$$W_{\text{eff}}(\Phi, M, \Lambda) = M\Phi^2 f\left(\frac{\Phi\Lambda}{M}\right), \tag{14.2}$$

where $f(t)$ must be holomorphic in t, and in particular $f(t)$ is regular as $t \to 0$ and $f(t)/t$ is regular as $t \to \infty$. Thus we must have $f(t) = a + bt$, something at most linear in t. Teh final condition hence fixes $a = \frac{1}{2}, b = \frac{1}{3}$. Hence

$$W_{\text{eff}}(\Phi, M, \Lambda) = \frac{M}{2}\Phi^2 + \frac{\Lambda}{3}\Phi^3 = W(\Phi, M, \Lambda). \tag{14.3}$$

We find that the effective potential is the same as the original potential. Finally, we freeze the superfields (M, Λ) to their VEVs (m, λ) by sending $\epsilon \to 0$ in the kinetic terms $\frac{1}{\epsilon}[\bar{M}M + \bar{\Lambda}\Lambda]$. The value of ϵ also cannot affect W_{eff} because we can always promote $1/\epsilon$ to a *real* superfield, and if supersymmetry is to hold, the superpotential can't depend on real superfields (only chiral superfields). More generally, the quantum superpotential is always independent of couplings appearing only in the Kähler potential $K(\Phi, \bar{\Phi})$.

In the end, we conclude that the effective superpotential is the same as it was classically–

$$W_{\text{eff}}(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi = W(\Phi) \tag{14.4}$$

and therefore receives no quantum corrections from perturbations in e.g. λ. In fact, with a bit more work this can be shown to be an exact, non-perturbative statement.

However, the Kähler potential *does* generically get quantum corrections. This is because the Kähler potential can depend on real superfields and is not guaranteed to be holomorphic; moreover, couplings from the superpotential can appear as couplings in the Kähler potential. In particular, the kinetic terms can receive corrections, so these can be nontrivial *wavefunction renormalization*, i.e.

$$\Phi_r = Z_\Phi^{1/2}\Phi \tag{14.5}$$

since e.g. we might have $\partial^\mu\bar\phi\partial_\mu\phi \to Z_\phi\partial^\mu\bar\phi\partial_\mu\phi = \partial^\mu\bar\phi_r\partial_\mu\phi_r$ after loop corrections. We might like to keep the kinetic term canonically normalized in terms of these ϕ_rs. In terms of the renormalized fields (for canonical kinetic terms), we then have

$$W_{\text{eff}}(\Phi_r) = \frac{m_r}{2}\Phi_r^2 + \frac{\lambda_r}{3}\Phi_r^3 \quad \text{where } m_r = Z_\Phi^{-1}m, \lambda_r = Z_\Phi^{-3/2}\lambda. \tag{14.6}$$

However, it's usually nicer to stick with the unrenormalized fields, since this makes the invariance of the superpotential under quantum corrections manifest.

Kähler geometry A *Kähler* manifold is a manifold $\mathcal{M}$ with three compatible structures: a Riemannian metric g, a (positive) symplectic form ω, and a complex structure J.

A 2-form $\omega \in \Omega^2(\mathcal{M})$ is *symplectic* if

- $d\omega = 0$ (i.e. $\omega = \omega_{ij}(x)dx^i \wedge dx^j$, then $\partial_{[i}\omega_{jk]} = 0$)
- ω is non-degenerate (i.e. for any vector field X, $\omega(X,Y) = 0 \forall$ vectors Y iff $X = 0$. Equivalently, ω_{ij} as an antisymmetric matrix is invertible)

A symplectic form is a natural candidate for a Poisson bracket structure.

A *almost complex structure* is a map $J : T\mathcal{M} \to T\mathcal{M}$ (i.e. on vectors in the tangent space) such that $J^2 = -\,\text{id}$. For example, on $\mathbb{R}^2$, $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is a basis of $T\mathcal{M}$, and we could choose

$$J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}. \tag{14.7}$$

Notice this feels a lot like multiplying by i, such that $J^2 = -1$.

We define the *holomorphic* and *antiholomorphic tangent bundles* on $\mathcal{M}$ (vector fields, if you like) as

$$T^{(1,0)}\mathcal{M} = \{X \in T\mathcal{M} \otimes \mathbb{C} : \frac{1}{2}(1 - iJ)X = X\}, \tag{14.8}$$

$$T^{(0,1)}\mathcal{M} = \{X \in T\mathcal{M} \otimes \mathbb{C} : \frac{1}{2}(1 + iJ)X = X\}. \tag{14.9}$$

That is, the holomorphic tangent bundle is the set of tangent vectors living in the $+i$ eigenspace of J, and the antiholomorphic tangent bundle is the set of tangent vectors in the $-i$ eigenspace.

An *almost complex structure* J is said to be integrable if $\forall X, Y \in T\mathcal{M} \otimes \mathbb{C}$, we have

$$\frac{(1+iJ)}{2}\left[\frac{(1-iJ)}{2}X, \frac{(1-iJ)}{2}Y\right] = 0. \tag{14.10}$$

That is, the Lie bracket of any two holomorphic vector fields is again holomorphic, so the tangent bundles decouple as we move around in the space. Taking the real and imaginary parts of these equations, we find that

$$N_J(X,Y) = -J^2([X,Y]) + J([JX,Y] + [X,JY]) - [JX,JY] = 0. \tag{14.11}$$

Here $N_J(X,Y)$ is known as the *Nijenhuis tensor*. Notice that J need not be a constant; in principle, these Lie brackets also differentiate the Js. However, the final result will turn out to not invole derivatives of the Xs and Ys. Moreover, this tensor is linear– for functions f, g, we have $N_J(fX, gY) = fgN_J(X,Y)$.

We now say that J is a complex structure iff it is an integrable almost complex structure. According to the Newlander-Nirenberg theorem, if any real manifold $\mathcal{M}$ has $N(X,Y) = 0$, then $\exists$ complex coordinates $x^i \to (z^a, \bar z^{\bar z})$ on any patch $U \subset \mathcal{M}$, and the transition functions between overlapping patches are purely holomorphic.

Note there may be different complex structures J on the same manifold, which may not be equivalent. Some manifolds admit no complex structures, some have uniquely one (e.g. the Riemann sphere), and others admit several inequivalent structures.

If J is a complex structure, we can split the tangent bundle up into holomorphic and antiholomorphic sectors,

$$T\mathcal{M} \otimes \mathbb{C} = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M} \tag{14.12}$$

globally. Similarly, we split $T^*\mathcal{M}\otimes\mathbb{C} = T^{*(1,0)}\mathcal{M}\oplus T^{*(0,1)}\mathcal{M}$ where $T_p^{*(1,0)}\mathcal{M}$ is the dual vector space to $T_p^{(1,0)}\mathcal{M}\forall p\in\mathcal{M}$. Hence if $\bar{\alpha}\in T^{*(0,1)}\mathcal{M}$ is an antiholomorphic one-form and $z\in T^{(1,0)}\mathcal{M}$ is in the holomorphic tangent bundle, then $\bar{\alpha}(z)=0$. Equivalently, $\bar{\alpha}\in T^{*(0,1)}\mathcal{M}$ iff $\bar{\alpha}=\bar{\alpha}_{\bar{a}}(z,\bar{z})d\bar{z}^{\bar{z}}$.

This structure extends– we can likewise split

$$\Omega^k(M,\mathbb{C}) = \bigoplus_{k=p+q}\Omega^{(p,q)}(\mathcal{M}), \tag{14.13}$$

the space of complex-valued k-forms, into spaces $\Omega^{(p,q)}(\mathcal{M})$ with p holomorphic indices and q antiholomorphic indices,

$$\eta(z,\bar{z}) = \eta_{a_1\ldots a_p\bar{b}_1\ldots\bar{b}_q}(z,\bar{z})dz^{a_1}\wedge\ldots\wedge dz^{a_p}\wedge d\bar{z}^{\bar{b}_1}\wedge\ldots\wedge d\bar{z}^{\bar{b}_q}. \tag{14.14}$$

We also have the exterior derivative operation $d:\Omega^k\to\Omega^{k+1}$, so on a $\mathbb{C}$-manifold this splits as $d=\partial+\bar{\partial}$ where $\partial:\Omega^{(p,q)}\to\Omega^{(p+1,q)}$ and $\bar{\partial}:\Omega^{(p,q)}\to\Omega^{(p,q+1)}$. For example, on $\mathbb{R}^2$,

$$d = dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y} = dz\frac{\partial}{\partial z}+d\bar{z}\frac{\partial}{\partial\bar{z}} \tag{14.15}$$

where $z=x+iy$. Also notice that

$$0=d^2=\partial^2+(\partial\bar{\partial}+\bar{\partial}\partial)+\bar{\partial}^2 \tag{14.16}$$

and we must have $\partial^2=0,\bar{\partial}^2=0,\partial\bar{\partial}+\bar{\partial}\partial=0$ separately since the form after each of these live in different spaces.

Lecture 15.

Tuesday, March 12, 2019

A complex manifold admits a structure of (p,q)-forms $\Omega^{p,q}(M)=\bigwedge^p T^{*(1,0)}\mathcal{M}\bigwedge^q T^{*(0,1)}M$. As we showed last time, the complex structure of the manifold and the nilpotency of the exterior derivative implies that $\partial^2=\bar{\partial}^2=\partial\bar{p}+\bar{p}\partial=0$ as operators.

A Kähler manifold has a symplectic form $\omega\in\Omega^2(\mathcal{M})$ that is compatible with J in the sense that

$$\omega(JX,JY)=\omega(X,Y)\quad\forall\text{ vector fields}X,Y. \tag{15.1}$$

This implies that ω actually lies in $\Omega^{1,1}(\mathcal{M})$,[5] so

$$\omega=\omega_{a,\bar{b}}(z,\bar{z})dz^a\wedge d\bar{z}^{\bar{b}}. \tag{15.2}$$

Given any such ω, we get a Hermitian metric for free, defined by

$$g(X,Y)=\omega(X,JY) \tag{15.3}$$

where $J:T\mathcal{M}\to T\mathcal{M}, J^2=-1$. (Recall that J is like multiplying by i.)

- We may check that this new metric g really is symmetric:

$$g(Y,X)=\omega(Y,JX)=-\omega(YX,Y)=\omega(JX,J^2Y)=\omega(X,JY)=g(X,Y) \tag{15.4}$$

  where we have used the antisymmetry of ω, the property that $J^2=-1$, and the fact that ω and J are compatible to explicitly show the symmetry of g.
- Moreover,

$$g(JX,JY)=\omega(JX,J^2Y)=-\omega(JX,Y)=\omega(Y,JX)=g(Y,X)=g(X,Y). \tag{15.5}$$

  Therefore g is compatible with J, which implies that g is indeed Hermitian.
- The metric g is positive iff ω is positive, $\omega(X,JX)>0$.

[5] Ω^2 decomposes into the direct sum $\Omega^{2,0}\oplus\Omega^{1,1}\oplus\Omega^{0,2}$.

Since $d\omega = 0$, on a $\mathbb{C}$ manifold we have $\partial\omega + \bar{\partial}\omega = 0$, where $\partial\omega \in \Omega^{2,1}$ and $\bar{\partial}\omega \in \Omega^{1,2}$. Hence the derivatives individually vanish, $\partial\omega = 0 = \bar{\partial}\omega$.

The complex form of the Poincaré lemma (i.e. if $d\alpha = 0$, then $\alpha = d\beta$ on any open $U \subset \mathcal{M}$) says that forms which are closed under ∂ and $\bar{\partial}$ locally must be exact, i.e. since $\partial\omega = \bar{\partial}\omega = 0$, $\exists K$ a real function on $\mathcal{M}$ (a $(0,0)$ form, if you like) such that

$$\omega = i\partial\bar{\partial}K. \tag{15.6}$$

Thus the metric is

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K \tag{15.7}$$

on any coordinate patch U, and we call this function K the *Kähler potential*. Notice it is defined up to transformations

$$K(z,\bar{z}) \to K(z,\bar{z}) + f(z) + \bar{f}(\bar{z}), \tag{15.8}$$

since the two derivatives will kill off the purely holomorphic and antiholomorphic contributions.

Example 15.9. The complex plane $\mathbb{C}^n$ is Kähler, with $K = \sum_{a=1}^n |z^a|^2$, where

$$g = \sum_{a=1}^n dz^a d\bar{z}^{\bar{a}} \tag{15.10}$$

$$\omega = i\sum_{a=1}^n dz^a \wedge d\bar{z}^{\bar{a}}. \tag{15.11}$$

Example 15.12. The complex projective plane $\mathbb{C}P^n$ is also Kähler, where $K = \ln\big(1 + \sum_{a=1}^n |z^a|^2\big)$, with the z^a defined on a $\mathbb{C}^n$ coordinate patch. The associated metric is called the Fubini-Study metric on $\mathbb{C}P^n$.

Note that on a Kähler manifold, the only non-vanishing pieces of the connection are

$$\Gamma^a_{bc} = \frac{1}{2} g^{ai}(\partial_b g_{ic} + \partial_c g_{ib} - \partial_i g_{bc}) \tag{15.13}$$

where $i = 1, \dots, 2n$ are real indices, $a = 1, \dots, n$ holomorphic indices. In fact, since the metric must have one holomorphic and one antiholomorphic index, we can WLOG replace i by an antiholomorphic index $\bar{d}$. Hence

$$\begin{aligned}\Gamma^a_{bc} &= \frac{1}{2} g^{a\bar{d}}(\partial_b g_{\bar{d}c} + \partial_c g_{\bar{d}b} - \partial_{\bar{d}} g_{bc})\\ &= \frac{1}{2} g^{a\bar{d}}(\partial_b \partial_{\bar{d}} \partial_c K + \partial_c \partial_{\bar{d}} \partial_b K)\\ &= g^{a\bar{d}}(\partial_b g_{c\bar{d}})\end{aligned}$$

and

$$\Gamma^{\bar{a}}_{\bar{b}\bar{c}} = g^{\bar{a}d}\partial_{\bar{b}} g_{\bar{c}d}. \tag{15.14}$$

One may check that all other components vanish by similar arguments. Hence the only non-vanishing pieces of the Levi-Civita connection are those components which have all holomorphic or all antiholomorphic indices.

Kähler manifolds and supersymmetry The general kinetic term in a supersymmetric nonlinear sigma model on $\mathbb{R}^{2/4}$ is $\int_{\mathbb{R}^{2/4}} K(\Phi, \bar{\Phi}) d^2x d^4\theta$. Notice this is defined only up to transformations $K \to K(\Phi, \bar{\Phi}) + f(\Phi) + \bar{f}(\bar{\Phi})$, as (up to total derivatives) such contributions will not survive $\int d^4\theta$. Performing the integrals gives

$$\begin{aligned}S_{\text{kin}}[\Phi, \bar{\Phi}] = \int_{\mathbb{R}^2} &-g_{a\bar{b}}\partial^\mu \phi^a \partial_\mu \bar{\phi}^{\bar{b}} + i g_{a\bar{b}} \bar{\psi}^{\bar{b}}_+ \nabla_- \psi^a_+ + i g_{a\bar{b}} \bar{\psi}^{\bar{b}}_- \nabla_+ \psi^a_-\\ &+ R_{a\bar{b}c\bar{d}} \psi^a_+ \bar{\psi}^b_+ \psi^c_- \bar{\psi}^{\bar{d}}_- + g_{a\bar{b}}(F^a - \Gamma^a_{cd}\psi^c_+ \psi^d_-)(\bar{F}^{\bar{b}} - \Gamma^{\bar{b}}_{\bar{e}\bar{f}} \bar{\psi}^{\bar{e}}_- \psi^{\bar{f}}_+),\end{aligned}$$

where $g_{a\bar{b}}(\phi, \bar{\phi}) = \partial_a \partial_{\bar{b}} K(\phi, \bar{\phi})$ is the Kähler metric and $\nabla_\mu \psi^a = \partial_\mu \psi^a_+ + \gamma^a_{bc} \partial_\mu \phi^b \psi^c_+$ where μ is a worldsheet index. One can explicitly compute all these terms, but we'll just make a plausibility argument. Recall that

$$\Phi(y^\pm, \theta^\pm) = \phi + \theta^+ \psi_+ + \theta^- \psi_- + \theta^+ \theta^- F, \tag{15.15}$$

with derivatives $\theta^+ \bar{\theta}^+ \partial_+, \theta^- \bar{\theta}^- \partial_-$ appearing as we expand $y^\pm x^\pm + i\theta^\pm \bar{\theta}^\pm$.

Eliminating the auxiliary fields $F, \bar{F}$ by their equations of motion $F^a = \Gamma^a_{bc}\psi^b_+\psi^c_-$, the remaining action is invariant under the following global symmetries:

- $\mathbb{C}$ coordinate transformations of the target space
- Kähler transformations
- SUSY transformations on the worldsheet (by construction)
- $U_V(1)$ (vector) transformations $\psi_\pm \to e^{i\beta}\psi_\pm, \bar{\psi}_\pm \to e^{-i\beta}\bar{\psi}_\pm, \phi \to \phi$.
- $U_A(1)$ (axial) transformations, $\psi_\pm \to e^{\pm i\alpha}\psi_\pm, \bar{\psi}_\pm \to e^{\mp i\alpha}\bar{\psi}_\pm, \phi \to \phi$
- Dilations (scale transformations) on the worldsheet (recall that in $d = 2$, $[\phi = 0]$, $[\psi_\pm] = 1/2$). If we like, we can expand the metric about a point such that we have some leading order canonical kinetic term behavior, and then the higher-order corrections away from that point represent interactions.

Thus at the classical level, this defines a supersymmetric CFT. Turning on a superpotential $\int W(\Phi), d^2\theta d^2x = \int \left(F^a\partial_a W - \frac{1}{2}\partial_a\partial_b W\psi^a_+\psi^b_-\right)d^2x$ breaks conformal invariance, since the couplings in $W(\Phi)$ can be dimensionful and therefore break conformal invariance.

These symmetries may or may not survive at the quantum level– we will see that symmetries of the action may not be present in the path integral, leading to anomalies. Closely related to that is the fact that in the quantum theory, we expect couplings to run. We'll try to show that these symmetries are anomaly-free if the manifold is not just Kähler but in fact Calabi-Yau.

Lecture 16.

Thursday, March 14, 2019

Consider the action

$$S[\phi,\psi] = \int \left[g_{a\bar{b}}h^{\mu\nu}\partial_\mu\phi^a\partial_\nu\bar{\phi}^{\bar{b}} + ig_{a\bar{b}}\bar{\psi}^{\bar{b}}\gamma^\mu\nabla_\mu\gamma^a + R_{a\bar{b}c\bar{d}}\psi^a_+\psi^c_-\bar{\psi}^{\bar{b}}_-\bar{\psi}^{\bar{d}}_+\right]\sqrt{h}dx^2 \tag{16.1}$$

with h a worldsheet metric. Classically, this is invariant under the scale transformations

$$h_{\mu\nu} \to \lambda^2 h_{\mu\nu}, \quad \gamma^\mu \to \lambda^{-1}\gamma^\mu, \quad \psi \to \lambda^{-1/2}\psi, \quad \lambda \in \mathbb{R}_{>0}. \tag{16.2}$$

However, quantum mechanically, there can be a non-zero β-function for the target space metric $g_{a\bar{b}}(\phi)$.

To understand this, first consider a purely bosonic nonlinear sigma model with Riemanninan target space:

$$S[\phi] = \frac{1}{2}\int g_{ij}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j dx^2. \tag{16.3}$$

Using Riemann normal coordiantes, $\phi^i = \phi^i_0 + \xi^i$, we have a flat metric plus corrections of order ξ^2 in terms of the curvature:

$$g_{ij}(\phi) = \delta_{ij} - \frac{1}{3}Rikjl(\phi_0)\xi^k\xi^l + O(\xi^3). \tag{16.4}$$

Then the action becomes

$$S[\xi] = \frac{1}{2}\int \delta_{ij}\partial^\mu\xi^i\partial_\mu\xi^j - \frac{1}{3}R_{ikjl}\xi^k\xi^l\partial_\mu\xi^i\partial^\mu\xi^j + \dots \tag{16.5}$$

The propagator in this theory from $\xi^i(x)$ to $\xi^j(y)$ is then

$$\int \frac{d^2k}{(2\pi)^2}\frac{e^{ik\cdot(x-y)}}{k^2}\delta^{ij}. \tag{16.6}$$

Note that this is logarithmically divergent as $k \to 0$ (which we'll deal with later by applying an IR cutoff). We also have a four-point vertex corresponding to

$$\frac{1}{6}\int R_{ikjl}(\phi_0)\xi^k\xi^l\partial_\mu\xi^i\partial^\mu\xi^j d^2x. \tag{16.7}$$

This vertex allows us to compute a one-loop correction to the propagator $\langle\xi^i(x)\xi^j(y)\rangle$. We get a contribution

$$\int \frac{d^2k}{(2\pi)^2}\frac{e^{ik\cdot(x-y)}}{k^2}\left[\delta^{ij} + \frac{1}{3}\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2}R^{ij}(\phi_0)\right] \tag{16.8}$$

Applying momenta cutoffs, we have

$$\int_{\mu<|p|<\lambda} \frac{d^2p}{(2\pi)^2}\frac{1}{p^2} = \frac{1}{2\pi}\int_\mu^\Lambda \frac{dp}{p} = \frac{1}{2\pi}\ln\left(\frac{\Lambda}{\mu}\right), \tag{16.9}$$

so the metric is renormalized and picks up a contribution from R^{ij} the Ricci tensor:

$$g_{ij}(\mu) = \delta_{ij} + \frac{1}{6\pi}R_{ij}\ln\left(\frac{\Lambda}{\mu}\right), \tag{16.10}$$

giving a beta function

$$\beta_{ij} = \frac{1}{6\pi}R_{ij} \tag{16.11}$$

proportional to the Ricci curvature.

- For $R_{ij} > 0$, we have $\beta_{ij} > 0$, so the model is asymptotically free– the curvature of the target space becomes less important at short distances on the worldsheet, so the theory makes sense in the UV.
- For $R_{ij} < 0$, the theory only makes sense as an effective theory (at least in this bosonic case), but becomes trivial in the IR.
- The interesting case is $R_{ij} = 0$, when the target space is Ricci flat/solves the vacuum Einstein equations. These are *conformally invariant* to (at least) one-loop accuracy.

Exactly the same calculations also hold in supersymmetric models. In particular there is non-zero running of the (Kähler) metric $g_{a\bar{b}} \to g_{a\bar{b}} + \#R_{a\bar{b}}$. Ricci flat Kähler manifolds are called *Calabi-Yau*. There was an expectation that Calabi-Yau manifolds might also enjoy some sort of non-renormalization theorem for the target space, but in fact these still receive quantum corrections starting at four loops. This poses an apparent problem for string theory, where we integrate over all conformal structures on the worldsheet. This doesn't really make sense if we can't construct conformally invariant structures on the worldsheet (i.e. the conformal invariance becomes part of the gauge symmetry and then by definition cannot be broken).

To address this problem, consider the correlation function $f(h,g) = \left\langle (\psi_-)^k(\bar{\psi}_+)^k \right\rangle$ for some $k \in \mathbb{Z}_{\geq 0}$. This vanishes unless there are exactly k zero modes of ψ_- and $\bar{\psi}_+$ since the ψ_- and $\bar{\psi}_+$s do not mix– $\nexists$ any Feynman graphs that can absorb these fermion insertions. By a zero mode, we mean a solution ψ_- such that $\nabla_+\psi_- = 0$ or $\bar{\nabla}\psi_- = 0$ on a Euclidean signature worldsheet. That is, there is a zero mode of ψ_- iff

$$H^0(\Sigma, \phi^* T^{(1,0)}\mathcal{M} \otimes S_-), \tag{16.12}$$

where H^0 indicates it is holomorphic on Σ, $T^{(1,0)}$ says it has a holomorphic target space index ψ^a, and S_- indicates it transforms as a spinor on Σ.

The index theorem (plus a vanishing theorem) says that if $\sigma = T^2$ the torus, then

$$\#(\psi_- \text{ z.m.s.}) = \int_{T^2}\phi^*(c_1(T^{(1,0)}N)) = \frac{1}{2\pi}\int_{T^2} R_{a\bar{b}}d\phi^a \wedge d\bar{\phi}^{\bar{b}} \in \mathbb{Z}_{\geq 0}. \tag{16.13}$$

Hence the number of zero modes sniffs out something topological about the target space.

It's also true that $\bar{\psi}_+$ is related to ψ_- by complex conjugation, so

$$\#\psi_- \text{ z. m.} = \#\bar{\psi}_+ \text{ z.m} \tag{16.14}$$

Now

$$f(h,g) = \left\langle \frac{(\psi_-)^k}{(\sqrt{\lambda})^k}\frac{(\bar{\psi}_+)^k}{(\sqrt{\lambda})^k} \right\rangle \lambda^k = f(\lambda^2 h, g)\lambda^k. \tag{16.15}$$

Also, $\psi_-, \bar{\psi}_+$ are invariant under the SUSY transformations $\bar{Q}_+, Q_-$, so we expect some form of localization. In fact, one can show that

$$f(h,g) = n_h e^{-\text{Area}(\Sigma, \phi^* g)} \tag{16.16}$$

related to the pullback of the target space metric, where $\text{Area}(T^2, \phi^* g) = \int_{T^2} g_{a\bar{b}}h^{\mu\nu}\partial_\mu\phi^a\partial_\nu\bar{\phi}^{\bar{b}}\sqrt{h}d^2x$.

Combining these, we see that

$$f(h,g) = f(\lambda^2 h, g)\lambda^k = n_{\lambda^2 h}e^{-(\text{Area}(g) - k\ln\lambda)} = f(\lambda^2 h, g') \tag{16.17}$$

where g' is a target metric such that $\text{Area}(T^2, \phi^* g') = \text{Area}(T^2, \phi^* g) - k\ln\lambda$.

However if $h_{\mu\nu} = \delta_{\mu\nu}$, then

$$\begin{aligned}
\mathrm{Area}(T^2, \phi^* g) &= i \int_{T^2} g_{a\bar{b}} (\partial_z \phi^a \partial_{\bar{z}} \bar{\phi}^{\bar{b}} + \partial_{\bar{z}} \phi^a \partial_z \bar{\phi}^{\bar{b}}) d^2 z \\
&= 2i \int_{T^2} g_{a\bar{b}} \partial_{\bar{z}} \phi^a \partial_z \bar{\phi}^{\bar{b}} d^2 z + i \int_{T^2} g_{a\bar{b}} (\partial_z \phi^a \partial_{\bar{z}} \bar{\phi}^{\bar{z}} - \partial_{\bar{z}} \phi^a \partial_z \bar{\phi}^{\bar{b}}) d^2 z \\
&= 2i \int_{T^2} g_{a\bar{b}} \partial_{\bar{z}} \phi^a \partial_z \bar{\phi}^{\bar{b}} d^2 z + \int_{T^2} \phi^* \omega
\end{aligned}$$

where $\omega_{a\bar{b}} = i g_{a\bar{b}} d\phi^a \wedge d\phi^{\bar{b}}$ is the Kähler form on N. In fact, the correlator localizes on holomorphic maps, so $\bar{\partial}_{\bar{z}} \phi^a = 0$ and Area $= \int_{T^2} \phi^* \omega$.

Thus the effect of rescalng the worldsheet metric $h_{\mu\nu} \to \lambda^2 h_{\mu\nu}$ means that

$$\int \phi^* \omega \to \int \phi^* \omega - \frac{i \ln \lambda}{2\pi} \int_{T^2} R_{a\bar{b}} d\phi^a \wedge d\bar{\phi}^{\bar{b}} \tag{16.18}$$

using the index theorem for k. That is, the Kähler class (up to exact pieces, i.e. the exterior derivative of something) is

$$[\omega] \to [\omega] - \frac{\log \lambda}{2\pi} [R], \tag{16.19}$$

and this result *is* exact in the SUSY theory.