87. Strong Cosmic Censorship in Asymptotically de Sitter Spacetimes

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Strong Cosmic Censorship and the End of Spacetime

In this essay we discuss the strong cosmic censorship conjecture, originally formulated by Penrose. This conjecture states that physically reasonable spacetimes are globally hyperbolic, so that given appropriate initial conditions, the Einstein equations are completely deterministic. Several recent studies of the conjecture have focused on the stability of the Cauchy horizon for black holes in asymptotically de Sitter spacetimes. We review these studies and explicitly show that the cosmological redshift effect due to the de Sitter background can prevent the blow-up and loss of regularity of perturbations near the Cauchy horizon, leading to a violation of strong cosmic censorship for the near-extremal Reissner-Nordström-de Sitter black hole.

Contents

1. INTRODUCTION

The strong cosmic censorship conjecture [1] is a statement about the limitations of classical general relativity. Stated simply, this conjecture asserts that *physically reasonable spacetimes are globally hyperbolic*, i.e. they admit a Cauchy surface on which suitable initial conditions can be specified so that a unique solution up to diffeomorphism consistent with the Einstein equations (the maximal Cauchy development) can be constructed for the entire spacetime manifold [2]. At heart, the conjecture is motivated by questions of the predictive power of the Einstein equations, considered as an initial value problem. Can astrophysical observers ever cross a Cauchy horizon? Does the deterministic nature of the Einstein equations ever break down? Are Cauchy horizons stable to perturbations? Strong cosmic censorship requires that the answer to all of these questions is "generically, no."

In this essay, we will first consider several different formulations of the strong cosmic censorship conjecture and justify why the modern formulation as posed by Christdoulou is most in the spirit of Penrose's orignal conjecture. We then introduce the quasinormal modes, a key tool in the study of perturbations to black hole spacetimes. We motivate the crucial result that knowing the decay of quasinormal modes near the event horizon is in fact sufficient to prove regularity of perturbations at the Cauchy horizon and thereby establish a violation of strong cosmic censorship. Finally, we characterize the parameter space of both the Reissner-Nordström-de Sitter (RNdS) and Kerr-de Sitter solutions in terms of the quasinormal modes and show that while the near-extremal RNdS solutions do violate all smooth formulations of cosmic censorship, no such violations are observed in the much more astrophysically relevant Kerr-de Sitter solutions.

2. DEFINING STRONG COSMIC CENSORSHIP

The strong cosmic censorship conjecture is usually stated as a constraint on the causal structure of spacetime with regards to Cauchy horizons. Recall that a Cauchy horizon represents the boundary of the domain of dependence for some initial data set. If Cauchy horizons are present in a spacetime, then by definition there are regions of spacetime which cannot be predicted from the initial conditions– determinism in our classical theory breaks down. If an observer could cross the Cauchy horizon, their future could not be determined uniquely based on their past. Strong cosmic censorship asserts that generically, *this does*

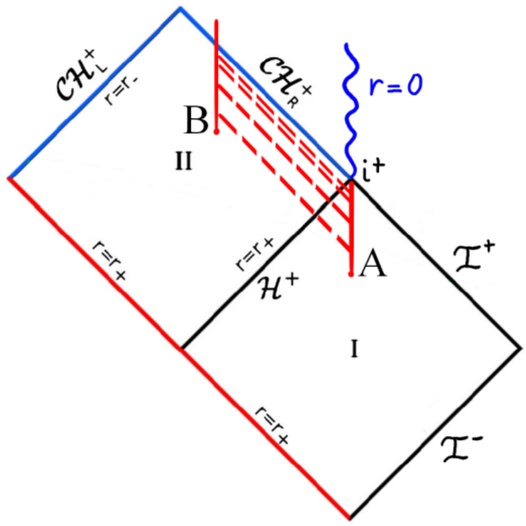

FIG. 1: A modified version of Figure 1 in [4]. A light clock A sends light pulses (dashed red lines) to an observer B at regular intervals of proper time, as measured by A. It takes infinite proper time for A to reach i^+, and so A sends infinitely many pulses to B. However, since B would cross the Cauchy horizon in finite proper time, B experiences an infinite proper time compression effect relative to A and must "see" all of A's pulses in finite proper time. This produces a singular energy density before B can actually reach the Cauchy horizon.

not happen.

Theorem 1 (Strong cosmic censorship (informal))**.** *Cauchy horizons are inherently unstable, and therefore cannot be crossed by astrophysical observers.*

That is, Cauchy horizons *can* exist under very specific circumstances (e.g. in the asymptotically flat Reissner-Nordström spacetime), but we need not take the new region on the other side of the Cauchy horizon too seriously. If Cauchy horizons are unstable, the region on the other side is in practice inaccessible and should not be considered as a physically meaningful part of spacetime.

This attempt captures the general spirit of the conjecture, but it is too vague. Let us try to be more precise. First, what do we mean by instability? The natural language to discuss instability is the language of perturbation theory. If we write down a solution with a Cauchy horizon and perturb the initial data (e.g. by breaking spherical symmetry), then generically,[1] the perturbed spacetime will not feature a Cauchy horizon. In the

[1] Reall has a more formal statement of what we mean by "generically." In his words, "if one introduces some measure on the space of geodesically complete, asymptotically flat initial data, strong cosmic

perturbed spacetime, the Cauchy horizon is precluded by some sort of singularity which makes it impossible for an astrophysical observer to cross the Cauchy horizon.

For instance, while the Reissner-Nordström solution in asymptotically flat space has a Cauchy horizon, it also suffers from a blue-shift instability making it sensitive to perturbation, as illustrated in the following example.

Example 2. Consider the asymptotically flat Reissner-Nordström solution, whose Penrose diagram is shown in Fig. 1. Suppose we have an observer B in the interior region of the black hole (region II) proceeding towards the Cauchy horizon at $r = r_-$. Falling freely, our observer can cross the Cauchy horizon in finite proper time according to their own clock. However, suppose we also place a clock A far away from the black hole in the asymptotically flat exterior region (region I), and this clock sends light pulses to our observer at regular intervals as measured in its own frame.

Since this spacetime is asymptotically flat, the clock will send an infinite number of pulses to our observer as it proceeds towards timelike infinity i^+, taking infinite proper time in its own frame to do so. However, by inspecting the Penrose diagram, we see that the observer B must receive this infinite number of pulses from the light clock A in *finite* proper time, specifically the proper time before B crosses the Cauchy horizon. In the observer's frame, the proper time between pulses gets shorter and shorter as the observer approaches the horizon, a blueshift which results in an infinite energy density at the Cauchy horizon itself and produces a spacetime singularity. The singularity (known as the Poisson-Israel mass inflation singularity [5]) destroys the smoothness of the metric at the Cauchy horizon, and therefore suggests that the Cauchy horizon in the asymptotically flat RN spacetime is unstable.

Having considered instability, we will next refine the idea of an "observer." In place of observers, we might study the behavior of the spacetime metric or any fields (electromagnetic, scalar, etc.) that may live in our spacetime. In the context of strong cosmic censorship, the instability of the Cauchy horizon means that the metric or any fields that may live in our spacetime cannot be extended across the Cauchy horizon while satisfying some differentiability condition (e.g. continuity or C^r smoothness). This is what we

censorship asserts that the maximal development is inextendible except for a set of initial data of measure zero." [3] In practice, this definition is difficult to work with, and we will instead study the blowup of perturbations near the Cauchy horizon in order to argue that the Cauchy horizon cannot be crossed for all "reasonable" spacetimes, i.e. ones subject to perturbation.

mean when we say the singularity prevents an "observer" from crossing the Cauchy horizon. Intuitively, if strong cosmic censorship holds, then our fields must become "badly behaved" (i.e. not smooth) as they approach the Cauchy horizon.

These refinements lead us to the following formulation of the strong cosmic censorship conjecture.

Theorem 3 (Strong cosmic censorship (C^r formulation))**.** *For a spacetime with a Cauchy horizon, consider an initial perturbation to the spacetime metric. Then generically, this perturbation cannot be extended to the Cauchy horizon in the smoothness class C^r.*

In this version, we have replaced terms like stability and observers with more careful statements about perturbations and smoothness at the Cauchy horizon. For concreteness, notice that the C^r formulation encompasses the C^0 formulation of strong cosmic censorship, which states that generically, the spacetime metric cannot be extended as a continuous function across the Cauchy horizon. The statement of the C^2 formulation is analogous, replacing the word "continous" with "twice-differentiable." Note the direction of the implication here– a violation of the C^r formulation implies a violation of all $C^{r'}$ formulations for $r' < r$ (e.g. C^1 false $\implies$ C^0 false). Conversely, if the C^r formulation is true, then all smoother formulations are also true (e.g. C^1 true $\implies$ C^2 true) but there is no implication for the formulations of lower smoothness.

One might hope that the C^0 formulation is true, since this would unambiguously answer the question of strong cosmic censorship by establishing that the metric is generically inextendible as a continuous function, and so spacetime simply comes to an end at the Cauchy horizon. However, in asymptotically flat space, the C^0 formulation has been shown by Dafermos and collaborators to be *false*– the metric can in fact be extended continuously across the Cauchy horizon even after perturbation [6, 7]. This result may be a bit surprising, but it is not quite sufficient to prove that our aforementioned "observers" can survive the trip. The equations of motion for our fields are second-order, so we might think that the metric should not just be continuous but C^2 smooth at the Cauchy horizon in order to solve the Einstein equations and pose a real threat to strong cosmic censorship. In the asymptotically flat case, the C^2 formulation does appear to be true [8].

In fact, this turns out to be too strong a requirement, due to the possibility of *weak solutions* to the Einstein equations. For a second-order equation of motion, we may multiply the equation by a compactly supported test function and integrate the second-derivative terms by parts to get a related equation which is only first-order. Functions which solve the new first-order equation for arbitrary test functions are then called "weak solutions." Weak solutions need not be as smooth as solutions to the original equations, but can still represent important physical phenomena like shocks in fluids. As shown by Klainerman et al. in [9], it is not the C^1 or C^2 smoothness of the metric but rather the L^2 norm of the curvature which determines the extendibility of the solution at the Cauchy horizon. This leads us to the so-called Christodoulou formulation of strong cosmic censorship [10].

Theorem 4 (Strong cosmic censorship (Christodoulou))**.** *For a spacetime with a Cauchy horizon, consider an initial perturbation to the spacetime metric. Then generically, this perturbation cannot be extended to the Cauchy horizon with locally square-integrable Christoffel symbols (i.e. the metric does not lie in the Sobolev space H^1_{loc}).*

This at last seems to be the correct formulation, in the sense that it presents a necessary and sufficient condition to guarantee the breakdown of the Einstein equations at the Cauchy horizon. If the Christodoulou formulation is true, then not even weak solutions to the Einstein equations can be constructed across the Cauchy horizon. Weak solutions seem to have the minimum regularity required to be considered as physically meaningful solutions, so that establishing the non-existence of weak solutions would unambiguously answer the question of strong cosmic censorship.

As a concluding note, there is a formulation of strong cosmic censorship due to Dafermos and Shlapentokh-Rothman (DSR) [11], which states that if the initial perturbation is allowed to lie in some minimum smoothness class (e.g. the initial data is already rough), then generically near the Cauchy horizon the perturbation will be less regular than the initial perturbation. That is, allowing rough initial data is enough to force the breakdown of the Einstein equations and thereby uphold strong cosmic censorship. The DSR formulation is interesting and may well be true, but it does not align as well with the spirit of Penrose's original conjecture. It is perhaps not so surprising that rough initial data can become ill-behaved and cannot be extended smoothly through the Cauchy horizon; it would be much more alarming if smooth initial data were extended smoothly through the Cauchy horizon, since this would force us to take the new region of spacetime on the other side seriously. In the rest of this essay, we shall focus on the validity of the C^r and Christodoulou formulations.

3. QUASINORMAL MODES

3.1. Definition of QNMs

In this section, we introduce a useful tool for the study of black hole perturbations. These are the *quasinormal modes* (QNMs), a set of quasistationary solutions to a dissipative wave equation describing perturbations which oscillate and decay with time.

For concreteness, let us consider a stationary, spherically symmetric $(3+1\text{D})$ black hole spacetime in Schwarzschild-like coordinates of the form

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega_2^2, \tag{3.1}$$

where

$$t \in (-\infty, +\infty), \quad r \in (r_+, +\infty), \tag{3.2}$$

with r_+ representing an event horizon ($F(r_+) = 0$), and the angular coordinates have their usual ranges $\theta \in [0, \pi], \phi \in [0, 2\pi]$. We will derive a set of solutions to the wave equation in this spacetime, imposing a particular set of boundary conditions at the event horizon and spatial infinity. We will also take these spacetimes to be asymptotically flat or de Sitter ($\Lambda \geq 0$). The boundary conditions for asymptotically AdS spacetimes are somewhat more subtle due to the timelike boundary [12], and we will not discuss them further.

Consider now a massless real scalar field Φ in this spacetime. It obeys the Klein-Gordon equation in curved space:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0. \tag{3.3}$$

Because the metric is stationary and spherically symmetric, we can decompose solutions of Eqn. 3.3 into spherical harmonics and a simple time dependence of the form $e^{-i\omega t}$ for some frequency $\omega \in \mathbb{C}$.

Definition 5. A *mode solution* is a solution to Eqn. 3.3 of the form

$$\Phi = \sum_{lm} \frac{\psi_{\omega lm}(r)}{r} Y_{lm}(\theta, \phi) e^{-i\omega t}, \tag{3.4}$$

where the Y_{lm} are spherical harmonics.

Crucially, the frequency ω need not be real. In general, ω is complex, so that instead of pure oscillating solutions, generically we will get solutions which oscillate *and* decay/grow with time. By substituting our mode solution into Klein-Gordon, one can show that the radial part of Φ obeys the following Schrödinger-like equation:

$$\frac{d^2\psi}{dr_*^2} + (\omega^2 - V_l)\psi = 0 \tag{3.5}$$

where

$$V_l \equiv F(r)\left(\frac{l(l+1)}{r^2} - \frac{F'}{r}\right) \tag{3.6}$$

defines an effective radial potential V_l in terms of the angular momentum number l and

$$dr_* \equiv \frac{dr}{F(r)} \tag{3.7}$$

defines a tortoise coordinate r_*. To reduce notational clutter, we have dropped the subscripts on $\psi_{\omega lm}$. Note that similar equations can be derived for perturbations of fields with different spins, masses, and charges, e.g. some of the equivalent effective potentials are given in Ref. [13] for pure de Sitter space. Similarly, coupled gravitational and electromagnetic perturbations are described by the Kodama-Ishibashi formalism [14], as is employed in the analysis of [15].

Definition 6. A *quasinormal mode* is a mode solution Φ obeying the boundary conditions

$$\Phi \sim \begin{cases} e^{-i\omega(t+r_*)} & \text{for } r_* \to -\infty \\ e^{-i\omega(t-r_*)} & \text{for } r_* \to +\infty. \end{cases} \tag{3.8}$$

These boundary conditions have a natural physical interpretation. For an asymptotically flat metric, the effective potential V_l vanishes at the event horizon and at spatial infinity. In an asymptotically de Sitter spacetime, V_l vanishes at the event horizon and at the cosmological horizon. The solutions to Eqn. (3.5) in these limits look like free wave solutions,

$$\psi \sim e^{\pm i\omega r_*} \text{ as } r_* \to \pm\infty. \tag{3.9}$$

Physically, these conditions just say that near the event horizon, solutions must be purely ingoing, and at the cosmological horizon, solutions must be purely outgoing.

The quasinormal modes form an infinite, discrete set of modes with some eigenfrequencies ω_{QNM}. Unlike the normal modes of oscillation, they do not form a complete set, so we cannot describe generic initial data as a sum of quasinormal modes. Instead, we should think of quasinormal modes as transient, *quasistationary* states which can be dynamically excited and then decay with time.

3.2. QNMs and regularity at the Cauchy horizon

Two natural questions now arise. What can these quasinormal modes teach us about the behavior of perturbations near the Cauchy horizon? And how do we go about computing the quasinormal mode eigenfrequencies? Resolving the second question is somewhat complicated– only a few spacetimes admit exact quasinormal mode solutions, though many others are amenable to numerical techniques. There are also a handful of useful analytic approximations to the quasinormal mode frequencies in certain limiting cases. We will revisit this topic in the next section after spending some time justifying why quasinormal modes are critical to our analysis.

In fact, the first question has a remarkably simple answer. The decay of quasinormal modes in the exterior region is related to the regularity of perturbations near the Cauchy horizon by the following theorem [15].

Theorem 7. *Let α be the spectral gap, i.e. the imaginary part of the quasinormal mode frequency corresponding to the slowest-decaying quasinormal mode, and let κ_- be the surface gravity of the Cauchy horizon. Then a linearized perturbation can be extended to the Cauchy horizon with regularity at least*

$$H^{1/2+\beta}, \quad \beta \equiv \alpha/\kappa_-. \tag{3.10}$$

This is a very useful result. Strictly, it must be proved separately for each kind of perturbation we wish to study, e.g. for massless scalar perturbations, for gravitoelectromagnetic perturbations, etc. However, once established, this theorem tells us that it is sufficient to study the decay of the quasinormal modes in order to establish regularity at the Cauchy horizon. If we can compute the spectral gap α, we can place a lower bound on the smoothness of perturbations at the Cauchy horizon. If *all* quasinormal modes in a region of parameter space have $\beta > 1/2$, this constitutes a violation of the Christodoulou formulation of strong cosmic censorship.

To prove Thm. 7, we will again consider scalar fields which admit a decomposition into mode solutions, as in Eqn. 3.4. Let us momentarily set aside the quasinormal mode boundary conditions and study these mode solutions in the interior region of the Reissner-Nordström-de Sitter (RNdS) spacetime, region II. Our analysis here will follow [15], which in turn rederives the results of Mellor and Moss in [16].

As in the exterior region I, we can define our tortoise coordinate r_* by $dr_* = dr/F(r)$

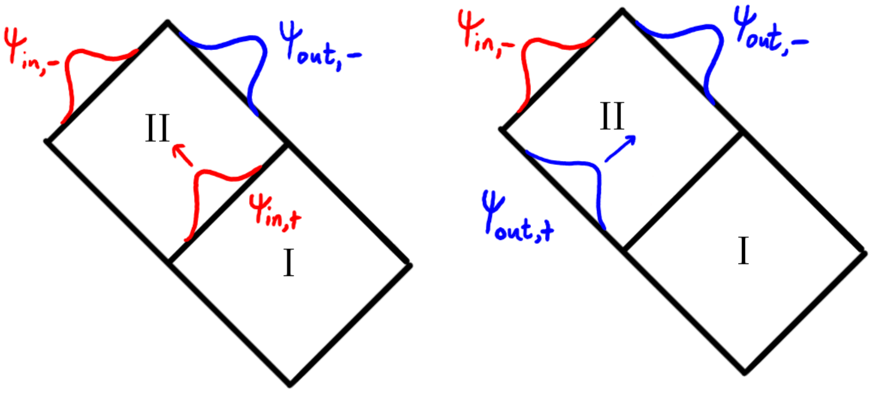

FIG. 2: Modes enter region II from the event horizon and scatter in the interior. In the left diagram, an in mode (red) at $\mathcal{H}_R^+$ scatters and is partially transmitted as $\psi_{\text{in},-}$ (red) and partially reflected as $\psi_{\text{out},-}$ (blue). On the right, the situation is flipped– an out mode (blue) at $\mathcal{H}_L^+$ scatters and is partially transmitted as $\psi_{\text{out},-}$ (blue) and partially reflected as $\psi_{\text{in},-}$ (red).

and write the Schrödinger-like equation governing the radial part of our mode solution:

$$\frac{d^2\psi}{dr_*^2} + (\omega^2 - V_l)\psi = 0, \tag{3.11}$$

with V_l again vanishing as $r_* \to \pm\infty$. In the black hole interior, region II, these limits now correspond to approaching the event horizon $\mathcal{H}^+$ ($r \to r_+, r_* \to -\infty$) and the Cauchy horizon $\mathcal{CH}^+$ ($r \to r_-, r_* \to +\infty$). Let us observe that near $\mathcal{H}^+$, we have two linearly independent solutions:

$$\psi_{\text{in},+} \sim e^{-i\omega r_*}, \quad \psi_{\text{out},+} \sim e^{+i\omega r_*}. \tag{3.12}$$

Here, the + subscript indicates that the behavior of these solutions is specified near the event horizon, $r = r_+$, and the terms "in" and "out" refer to the fact that these modes correspond to ingoing and outgoing solutions with respect to the ingoing and outgoing Eddington-Finkelstein coordinates v, u,[2] so that for these solutions,

$$\Phi_{\text{in},+} \sim e^{-i\omega(t+r_*)} = e^{-i\omega v}, \quad \Phi_{\text{out},+} \sim e^{-i\omega(t-r_*)} = e^{-i\omega u}. \tag{3.13}$$

Similarly, two linearly independent solutions can be constructed near the Cauchy horizon $\mathcal{CH}^+$ at $r = r_-$:

$$\psi_{\text{in},-} \sim e^{-i\omega r_*}, \quad \psi_{\text{out},-} \sim e^{+i\omega r_*}. \tag{3.14}$$

[2] We emphasize that the use of "in" and "out" should always be understood with reference to the ingoing and outgoing coordinates v, u, which are sensibly named in region I but are potentially more confusing in region II, where there exists both an "out" mode $\psi_{\text{out},+}$ entering region II and an "in" mode $\psi_{\text{in},-}$ leaving region II.

In particular, note that the modes near the event horizon are related to the modes near the Cauchy horizon. As shown in Fig. 2, for $\psi_{\text{in},+}$ (the in mode at $\mathcal{H}_R^+$) some fraction $\mathcal{A}(\omega)$ is transmitted to $\mathcal{CH}_L^+$ as $\psi_{\text{in},-}$ and some fraction $\mathcal{B}(\omega)$ is reflected to $\mathcal{CH}_R^+$ as $\psi_{\text{out},-}$. An equivalent statement holds for $\psi_{\text{out},+}$, with some corresponding transmission and reflection coefficients $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$.

By time reversal symmetry, the transmission and reflection coefficients for modes entering region II from $\mathcal{H}_R^+$ are the same as the transmission and reflection coefficients for waves entering region II from $\mathcal{CH}_L^+$ and $\mathcal{CH}_R^+$ respectively and being scattered to $\mathcal{H}_R^+$, and a similar statement holds for modes entering at $\mathcal{H}_L^+$. This statement can be summarized by the following relations:

$$\psi_{\text{out},+} = \mathcal{A}(\omega)\psi_{\text{out},-} + \mathcal{B}(\omega)\psi_{\text{in},-} \tag{3.15a}$$

$$\psi_{\text{in},+} = \tilde{\mathcal{A}}(\omega)\psi_{\text{in},-} + \tilde{\mathcal{B}}(\omega)\psi_{\text{out},-}, \tag{3.15b}$$

which totally characterize the behavior of modes near the Cauchy horizon given initial conditions near the event horizon.

Let us suppose the initial data on $\mathcal{H}_L^+$ and $\mathcal{H}_R^+$ are wavepackets with Fourier transforms $Z(\omega), \tilde{Z}(\omega)$ respectively. Thus

$$\Phi|_{\mathcal{H}_L^+} = \int d\omega\, e^{-i\omega u} Z(\omega) Y_{lm}(\theta,\phi) \tag{3.16a}$$

$$\Phi|_{\mathcal{H}_R^+} = \int d\omega\, e^{-i\omega v} \tilde{Z}(\omega) Y_{lm}(\theta,\phi) \tag{3.16b}$$

We can now match these boundary conditions to our linearly independent solutions $\psi_{\text{in},+}, \psi_{\text{out},+}$ at the horizon, so that

$$\Phi_L = \int d\omega\, e^{-i\omega t} \psi_{\text{out},+}(\omega, r) Z(\omega) Y_{lm}(\theta,\phi), \tag{3.17a}$$

$$\Phi_R = \int d\omega\, e^{-i\omega t} \psi_{\text{in},+}(\omega, r) \tilde{Z}(\omega) Y_{lm}(\theta,\phi) \tag{3.17b}$$

and the solution in region II is then

$$\begin{aligned}\Phi &= \Phi_L + \Phi_R \\ &= \int d\omega\, e^{-i\omega t}\Big[\psi_{\text{out},+}(\omega, r) Z(\omega) + \psi_{\text{in},+}(\omega, r)\tilde{Z}(\omega)\Big] Y_{lm}(\theta,\phi).\end{aligned} \tag{3.18}$$

Moreover, we can now decompose this solution in terms of in and out modes at the Cauchy horizon using the relations of Eqns. 3.15a and 3.15b:

$$\Phi = \Phi_{\text{out}} + \Phi_{\text{in}}, \tag{3.19}$$

where

$$\Phi_{\text{out}} = \int d\omega\, e^{-i\omega t}\Big[\mathcal{A}(\omega)Z(\omega) + \tilde{\mathcal{B}}(\omega)\tilde{Z}(\omega)\Big]\psi_{\text{out},-}(\omega,r)Y_{lm}(\theta,\phi), \tag{3.20a}$$

$$\Phi_{\text{in}} = \int d\omega\, e^{-i\omega t}\Big[\mathcal{B}(\omega)Z(\omega) + \tilde{\mathcal{A}}(\omega)\tilde{Z}(\omega)\Big]\psi_{\text{in},-}(\omega,r)Y_{lm}(\theta,\phi). \tag{3.20b}$$

Notice that Φ_{in} is smooth at the "left" part of the Cauchy horizon $\mathcal{CH}^+_L$, while Φ_{out} is smooth at the "right" Cauchy horizon $\mathcal{CH}^+_R$. Our goal throughout this calculation is to determine the regularity of modes near the Cauchy horizon, so let us study the smoothness of the in modes Φ_{in} at $\mathcal{CH}^+_R$.

Near the Cauchy horizon, $r_* \to +\infty$ and $\psi_{\text{in},-} \to e^{-i\omega r_*}$, so

$$\Phi_{\text{in}} \approx \int d\omega\, e^{-i\omega v}\Big[\mathcal{B}(\omega)Z(\omega) + \tilde{\mathcal{A}}(\omega)\tilde{Z}(\omega)\Big]Y_{lm}(\theta,\phi). \tag{3.21}$$

Taking a derivative with respect to the Kruskal-like coordinate $V_- \equiv -e^{-\kappa_- v}$, we find that

$$\partial_{V_-}\Phi_{\text{in}} \approx (-V_-)^{-1}\int d\omega\, e^{-i\omega v}\mathcal{F}(\omega)Y_{lm}(\theta,\phi), \tag{3.22}$$

where we have defined

$$\mathcal{F} \equiv -i\omega\Big[\mathcal{B}(\omega)Z(\omega) + \tilde{\mathcal{A}}(\omega)\tilde{Z}(\omega)\Big]. \tag{3.23}$$

The smoothness of Φ_{in} at $\mathcal{CH}^+_R$ therefore depends on the integral in Eqn. 3.22, and the value of this integral (which we can evaluate using contour integration) depends on the analyticity of $\mathcal{F}(\omega)$. Note that near $\mathcal{CH}^+_R$, the coordinate v goes to $+\infty$ (so that $\mathcal{CH}^+_R$ lies at $V_- = 0$). Because of the factor $e^{-i\omega v}$ in the integral, we will need to close the contour in the lower half-plane, and our interest will therefore be in the poles of $\mathcal{F}$ with negative imaginary components. As we will show, studying the pole structure of $\mathcal{F}$ in fact leads us back to the quasinormal mode frequencies. This result is summarized in the following lemma:

Lemma 3.24. *The poles of $\mathcal{F}(\omega)$ lie at positive integer multiples of $i\kappa_+$, negative integer multiples of $i\kappa_-$, and the quasinormal mode frequencies ω_{QNM}.*

To study the analyticity of $\mathcal{F}(\omega)$ and prove our lemma, we will rewrite $\mathcal{B}(\omega)$ and $\tilde{\mathcal{A}}(\omega)$ in terms of Wronskians of our solutions. Suppose we have two functions $f(r_*), g(r_*)$ which solve an equation of the form

$$\frac{d^2 f}{dr_*} - q(r_*)f = 0, \tag{3.25}$$

as is the case with our Eqn. 3.11. Then the *Wronskian* is defined to be

$$W[f,g] \equiv f'(r_*)g(r_*) - g'(r_*)f(r_*). \tag{3.26}$$

Notice that

$$\frac{d}{dr_*}W[f,g] = f''g - g''f = qfg - qfg = 0 \tag{3.27}$$

since f and g each solve 3.25, so this implies that the Wronskian of two solutions is constant in r_*. A bit of algebra reveals that

$$\tilde{\mathcal{A}}(\omega) = \frac{W[\psi_{\text{in},+},\psi_{\text{out},-}]}{W[\psi_{\text{in},-},\psi_{\text{out},-}]} = -\frac{W[\psi_{\text{in},+},\psi_{\text{out},-}]}{2i\omega}, \tag{3.28}$$

where the second equality comes from evaluating the denominator at $r_* \to +\infty$ (since $\psi_{\text{in},-}$ and $\psi_{\text{out},-}$ reduce to $e^{-i\omega r_*}$ and $e^{+i\omega r_*}$ near the Cauchy horizon and we've just shown that the Wronskian is constant in r_*). Similarly, the second coefficient is

$$\mathcal{B}(\omega) = \frac{W[\psi_{\text{out},+},\psi_{\text{out},-}]}{W[\psi_{\text{in},-},\psi_{\text{out},-}]} = -\frac{W[\psi_{\text{out},+},\psi_{\text{out},-}]}{2i\omega}. \tag{3.29}$$

Hence

$$\mathcal{F} = \frac{1}{2}\Big[W[\psi_{\text{out},+},\psi_{\text{out},-}]Z(\omega) + W[\psi_{\text{in},+},\psi_{\text{out},-}]\tilde{Z}(\omega)\Big]. \tag{3.30}$$

From here, we can cite results about the analyticity of $\psi_{\text{in},\pm}$ and $\psi_{\text{out},\pm}$ as originally derived by Chandrasekhar and Hartle in [17]. As it turns out, $\psi_{\text{in},+}(\omega,r)$ has simple poles at negative integer multiples of $i\kappa_+$, $\psi_{\text{out},+}$ has simple poles at positive integer multiples of $i\kappa_+$, and $\psi_{\text{in},-}$ has simple poles at negative integer multiples of $i\kappa_-$. This almost completely describes the pole structure of $\mathcal{F}$.

However, note that the initial data on $\mathcal{H}_R^+$ (as described by the Fourier transform $\tilde{Z}(\omega)$) is not completely generic, since it is determined by the initial conditions in the exterior region, region I. If we now suppose that initial data is specified on $\mathcal{H}^-$ and $\mathcal{H}_c^-$, the past event horizon and past cosmological horizon respectively, then we can perform a similar scattering analysis in region I to deduce the form of $\tilde{Z}$.

More concretely, suppose the initial data on $\mathcal{H}^-$ can be written as

$$\Phi|_{\mathcal{H}^-} = \int d\omega\, e^{-i\omega u}X(\omega)Y_{lm}(\theta,\phi) \tag{3.31}$$

and the initial data on $\mathcal{H}_c^-$ can similarly be written as

$$\Phi|_{\mathcal{H}_c^-} = \int d\omega\, e^{-i\omega v}\tilde{X}(\omega)Y_{lm}(\theta,\phi). \tag{3.32}$$

Then repeating the scattering analysis in region I with these boundary conditions and writing the corresponding transmission and reflection coefficients in terms of Wronskians (as is done in [15]) yields

$$\tilde{Z} = \left(-\frac{2i\omega}{W[\psi_{\text{in},+}, \psi_{\text{out},c}]}\right)\tilde{X}(\omega) + \left(\frac{W[\psi_{\text{out},+}, \psi_{\text{out},c}]}{W[\psi_{\text{in},+}, \psi_{\text{out},c}]}\right)X(\omega). \tag{3.33}$$

In terms of Z, X, and $\tilde{X}$, which we may now take to be generic entire functions, we therefore have

$$\mathcal{F} = \frac{1}{2}\left[W[\psi_{\text{out},+}, \psi_{\text{out},-}]Z(\omega) + \frac{W[\psi_{\text{in},+}, \psi_{\text{out},-}]}{W[\psi_{\text{in},+}, \psi_{\text{out},c}]}\left(-2i\omega\tilde{X}(\omega) + W[\psi_{\text{out},+}, \psi_{\text{out},c}]X(\omega)\right)\right]. \tag{3.34}$$

What poles does $\mathcal{F}$ have in the lower half-plane? From the first term, $\psi_{\text{out},-}$ contributes poles at negative integer multiples of $i\kappa_-$. The term with the ratios of Wronskians will also have some poles, but some of them will cancel. We can see this cancellation by a toy calculation: consider three functions, $f(x), g(x), h(x)$ where f has a simple pole at some x_0 and g, h are both regular at x_0. That is, near x_0, $f \sim \frac{f_0}{x-x_0}$. Then

$$\frac{W[f,g]}{W[f,h]} = \frac{f'g - fg'}{f'h - fh'} \sim \frac{-g(x_0)\frac{f_0}{(x-x_0)^2} - \frac{f_0}{x-x_0}g'(x_0)}{-h(x_0)\frac{f_0}{(x-x_0)^2} - \frac{f_0}{x-x_0}h'(x_0)} = \frac{-g(x_0)f_0 - f_0 g'(x_0)(x-x_0)}{-h(x_0)f_0 - f_0 h'(x_0)(x-x_0)}, \tag{3.35}$$

which is now perfectly regular as $x \to x_0$. For our function $\mathcal{F}$, the poles from $\psi_{\text{in},+}$ will cancel in this way, as will the poles from $\psi_{\text{out},c}$. The poles from $\psi_{\text{out},+}$ lie in the upper half-plane and will not contribute to our contour integral. The only remaining possibility for singular behavior is that $W[\psi_{\text{in},+}, \psi_{\text{out},c}]$ vanishes at some values of ω.

However, notice that if $W[\psi_{\text{in},+}, \psi_{\text{out},c}] = 0$, this tells us that the mode solutions which are ingoing at the event horizon and the solutions which are outgoing at the cosmological horizon are in fact linearly dependent. But this is simply the quasinormal mode boundary conditions of Eqn. 3.8, written in the language of Green's functions.

Hence the poles of $\mathcal{F}$ in the lower half-plane lie at negative integer multiples of $i\kappa_-$ and at the quasinormal mode frequencies ω_{QNM}, as illustrated in Fig. 3. We can then deform the contour of integration to place a bound on the integral $\int d\omega\, e^{-i\omega v}\mathcal{F}(\omega)Y_{lm}(\theta,\phi)$. In particular, we can shift the contour down off the real axis until just above the slowest-decaying quasinormal mode, integrating over ω with $\text{Im}(\omega) = -\alpha + \epsilon$. It is possible that in doing so, we will cross a (finite) number of poles at multiples of $-i\kappa_-$ on the imaginary axis, but these poles contribute an overall v dependence of $e^{-n\kappa_- v} = -(V_-)^n$ to our integral, so that referring back to Eqn. 3.22, these poles contribute terms to $\partial_{V_-}\Phi$ which go as $(V_-)^{n-1}$, which is smooth for all $n \in \mathbb{N}$ as $V_- \to 0$.

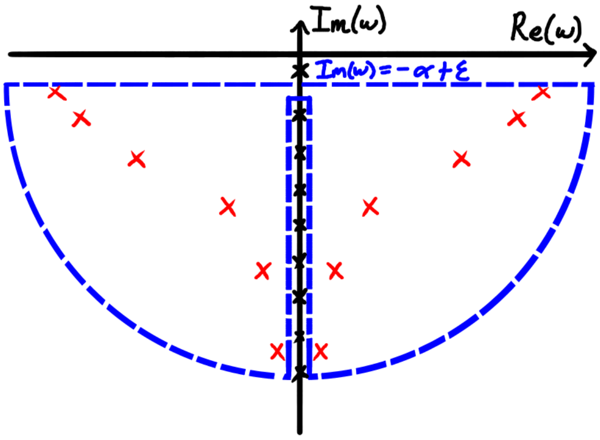

FIG. 3: The contour (dashed blue curve) for the integral $\int d\omega\, e^{-i\omega v}\mathcal{F}(\omega)Y_{lm}(\theta,\phi)$ in the complex ω plane. Poles at $-in\kappa_-, n \in \mathbb{N}$ are marked as black crosses on the imaginary axis, while QNM frequencies are red crosses. The integration contour has been pushed down off the real axis to just above the slowest-decaying QNM frequency, which has imaginary part $-i\alpha$ where α is the spectral gap.

Along our new contour, the integral goes as

$$\int_{-\infty}^{\infty} d\omega_r\, e^{-iv(\omega_r - i\alpha)}\mathcal{F}(\omega)Y_{lm}(\theta,\phi) \sim e^{-\alpha v} = (-V_-)^{\alpha/\kappa_-} \tag{3.36}$$

and therefore

$$\partial_{V_-}\Phi \sim (V_-)^{\beta-1}, \tag{3.37}$$

where $\beta = \alpha/\kappa_-$ as originally stated in Eqn. 3.10.

Let us discuss the importance of this calculation. Since the (right) Cauchy horizon lies at $V_- = 0$, we see that for $\beta < 1$, $\partial_{V_-}\Phi$ diverges as $V_- \to 0$, representing a blowup of curvature. Such a configuration upholds the C^1 formulation of strong cosmic censorship. If $\beta < 1/2$ for all regions of parameter space, then $\partial_{V_-}\Phi$ will not be square-integrable and thus the field cannot be extended even as a weak solution, in support of the Christodoulou formulation of strong cosmic censorship. Conversely, if $\beta = r$, then from Eqn. 3.37, we see that the first r derivatives of our field are smooth at the Cauchy horizon and the field is therefore extendible in C^r, establishing a violation of the C^r formulation of strong cosmic censorship.

Moreover, equivalent results can be derived to describe other fields– for example, Ref. [15] performs the analysis for linear coupled gravitational and electromagnetic perturbations, finding the same $\beta > 1/2$ condition for gravitoelectromagnetic quasinormal modes to be extendible as a weak solution at the Cauchy horizon, while Ref. [4] makes a similar argument for linearized metric perturbations in Kerr-de Sitter. In [18], Hintz and Vasy show that in general, solutions u to the wave equation $\Box_g u = 0$ with smooth initial data on RNdS and Kerr-de Sitter spacetimes of dimension ≥ 4 can be decomposed near the Cauchy horizon as $u = u_0 + u'$ for some constant $u_0 \in \mathbb{C}$ and where u' lies in the Sobolev space $H^{1/2+\alpha/\kappa_-}$, with α the spectral gap. This conclusion was further supported by the nonlinear analysis of Costa et al. in [19].

Taken together, these results suggest that a firm understanding of quasinormal modes will be key to establishing any violations of strong cosmic censorship in RNdS and Kerr-de Sitter spacetimes. In the next section, we will rederive analytical results about the quasinormal modes of these spacetimes and compare them to recent numerical work, arguing that not only is the Christodoulou formulation of strong cosmic censorship violated for the massless scalar field in RNdS, but in fact all C^r formulations are violated for the gravitoelectromagnetic quasinormal modes of the near-extremal RNdS spacetime.

4. THREE FAMILIES

4.1. Redshift, blueshift

Having introduced the quasinormal modes, what can we say about them in different regions of parameter space for our problem? In both the RNdS and Kerr-de Sitter solutions, there are two competing influences on the behavior of quasinormal modes near the event horizon. These are the *redshift effect* due to the positive cosmological constant of the background spacetime and the *blueshift effect* due to the gravity of the black hole itself. Depending on which of these effects wins out, perturbations will either grow or decay near the event horizon, which in turn determines their regularity near the Cauchy horizon.

Naturally, we would like to parametrize these two effects to understand their relative impacts. It is convenient to write the redshift effect in terms of a parameter

$$y_+ \equiv r_+/r_c, \tag{4.1}$$

since the radius of the cosmological horizon r_c is directly related to the size of the cosmological constant (e.g. in pure de Sitter this is just the de Sitter radius, $r_c^2 = \frac{3}{\Lambda}$). Large values of Λ therefore have the effect of bringing r_c in closer to the event horizon and increasing the redshift of perturbations as $y_+ \to 1$.

Conversely, the blueshift effect is governed by how close the black hole is to extremality, in terms of $Q/Q_{\rm ext}$ for the RNdS solution and $a/a_{\rm ext}$ for the Kerr-dS solution. That is, $Q_{\rm ext}$ and $a_{\rm ext}$ are the limiting values of Q and a where $r_+ = r_-$ for the RNdS and Kerr-de Sitter solutions, respectively. Recall that in the asymptotically flat Reissner-Nordström solution, as $Q \to Q_{\rm ext} = M$, the inner and outer horizons coincide, $r_+ = r_- = M$, and the surface gravity at the event horizon is

$$\kappa_+ = \frac{1}{2}|F'(r_+)| = \frac{1}{2}\left|\frac{2M}{r_+^2} - \frac{2Q^2}{r_+^3}\right| \to \frac{1}{2}\left|\frac{2M}{M^2} - \frac{2M^2}{M^3}\right| = 0. \tag{4.2}$$

A similar calculation shows that the surface gravity of the asymptotically flat extremal Kerr solution also vanishes as $a \to a_{\rm ext} = M$. As the surface gravity of the black hole gets weaker, we might expect the blueshift effect to also become weaker.

Importantly, these two parameters entirely characterize the problem. While both the RNdS and Kerr-de Sitter spacetimes represent three-parameter families of solutions (Λ, M, and either Q or a), we can always fix a mass scale (e.g. by choosing Λ or M) and normalize the other parameters to this scale, leaving two free parameters.

To sum up, we expect near-extremal black holes to have relatively weaker blueshift instabilities, while black holes in spacetimes with large cosmological constants Λ will have stronger damping redshift effects. These two effects are in direct correspondence with the two parameters describing the RNdS and Kerr-de Sitter families of solutions, i.e. the cosmological constant and the charge or angular momentum, respectively.

The question of strong cosmic censorship therefore reduces to considering different perturbations (e.g. of various spins, charges, and masses) to these black hole spacetimes and studying their behavior near the Cauchy horizon throughout our two-dimensional parameter space.

4.2. Photon sphere modes

The first family of quasinormal modes we will consider is the set of *photon sphere modes*. These modes dominate (i.e. are the relevant, slowest-decaying modes) for much of the parameter space where Λ is non-negligible and Q is still far from extremality. In addition, the photon sphere modes are believed to be only weakly dependent on the spin of the perturbation [20], and are therefore useful for generic quasinormal mode analyses. In this subsection, we will show that the quasinormal frequencies of the photon sphere modes are precisely related to the Lyapunov exponents (a kind of stability parameter) of unstable circular null geodesics around the RNdS black hole. Having proven this result, we will analytically calculate these unstable circular null geodesics, determine their corresponding Lyapunov exponents, and thereby derive an analytic approximation for the quasinormal frequencies of the photon sphere modes.

Our first result about the photon sphere modes is summarized in the following theorem [21].

Theorem 8. *In the geometric optics limit, $l \to \infty$, the quasinormal mode frequencies of a black hole spacetime are given by*

$$\omega_{\mathrm{WKB}} = l\Omega_0 - i(n + 1/2)|\lambda|, \tag{4.3}$$

where Ω_0 is the coordinate velocity of an unstable null circular orbit about the black hole, λ is the principal Lyapunov exponent of this orbit, and n and l are non-negative integers.

Notice that $\mathrm{Im}(\omega_{\mathrm{WKB}})$ is independent of l in this limit, so we can determine the slowest decaying mode (i.e. the spectral gap) by taking the fundamental frequency with $n = 0$.

Physically, the geometric optics limit tells us to study solutions which oscillate rapidly over a slowly changing background potential. Recall that quasinormal modes solve a Schrödinger-like equation, Eqn. 3.5, reproduced here:

$$\frac{d^2\psi}{dr_*^2} + (\omega^2 - V_l)\psi = 0 \tag{4.4}$$

with a potential

$$V_l \equiv F(r)\left(\frac{l(l+1)}{r^2} - \frac{F'}{r}\right) \tag{4.5}$$

In this limit, our Schrödinger equation, Eqn. 4.4, now becomes

$$\frac{d^2\psi}{dr_*^2} + U(r)\psi = 0 \text{ where } U \equiv \omega^2 - F(r)\frac{l^2}{r^2}. \tag{4.6}$$

If we were to draw U as a function of r_*, we would expect something that asymptotes to ω^2 at $r_* \to \pm\infty$ (where $F(r) \to 0$) and dips in the middle. We previously wrote down some free solutions $\psi \sim e^{\pm i\omega r_*}$ to Eqn. 4.4 in these asymptotic regions where $U \approx \omega^2$, and we'd now like to patch them together in the critical region where U is changing rapidly, i.e. near a classical turning point. The WKB approximation (perhaps familiar from undergraduate quantum mechanics) provides us a means of doing so.

Let r_0 be the value of r such that $\left.\frac{dU}{dr_*}\right|_{r=r_0} = 0$. The WKB approximation tells us that if we expand U about this extremum to order r_*^2, we can write down exact solutions for

ψ (in terms of parabolic cylinder functions) valid in this patching region and obeying the QNM boundary conditions, provided that U obeys a quantization condition [12, 21]

$$\frac{U(r_0)}{\sqrt{2\left.\frac{d^2U}{dr_*^2}\right|_{r=r_0}}} = i(n+1/2), \quad n = 0, 1, 2, \ldots$$

Since U is given by Eqn. 4.6 in terms of ω, F, and l, we can solve this for the quasinormal mode frequencies ω to find

$$\omega_{\text{WKB}} = l\sqrt{\frac{F_0}{r_0^2}} - \frac{i(n+1/2)}{\sqrt{2}}\sqrt{\frac{r_0^2}{F_0}\frac{d^2}{dr_*^2}\left[\frac{F(r)}{r^2}\right]\bigg|_{r=r_0}} \tag{4.7}$$

where $F_0 = F(r_0)$.

In principle, this formula now gives us all the quasinormal mode frequencies in terms of n, l, and black hole parameters like M and Q, at least in the large l limit. But this formula as written isn't very physically meaningful. In the next calculation, we will show that the quasinormal mode frequencies in the geometric optics limit are simply related to the physical parameters governing unstable null circular orbits in the black hole spacetime.

In most black hole spacetimes, there is an null circular orbit which we call the *photon sphere*. Generally, it is unstable. Given a stationary, spherically symmetric metric, we can use the usual Euler-Lagrange procedure to compute the corresponding conserved quantities of the null orbits ($E = F\dot{t}$ and $J = r^2\dot{\theta}$, the energy and angular momentum) and write down the equation of motion for the r coordinate,

$$\dot{r}^2 - V_r = 0 \text{ with } V_r \equiv E^2 - F(r)\frac{J^2}{r^2}. \tag{4.8}$$

For these to be circular orbits, we require that $\dot{r} = 0$ and $\ddot{r} = 0$, so null circular orbits satisfy

$$V_r(r_s) = 0, \quad V_r'(r)|_{r=r_s} = 0, \tag{4.9}$$

where r_s indicates the radius of the null circular orbit.

However, we now notice something interesting. V_r has the same form as the "potential" function U we defined in Eqn. 4.6, the Schrödinger equation after taking the eikonal limit. In particular, U and V_r will have an extremum at the same value of r. This is because $\frac{d}{dr} = F\frac{d}{dr*}$ by definition, so away from zeroes of F,

$$\frac{dU}{dr} = F\frac{dU}{dr_*} = 0, F \neq 0 \implies \frac{dU}{dr_*} = 0. \tag{4.10}$$

Therefore the r_0 which extremizes U is actually the radius of the photon sphere, i.e. $r_s = r_0$.

Moreover, if we set $V_r(r_s) = 0$, we get an expression relating E and J,

$$b \equiv \frac{J}{E} = \sqrt{\frac{r_s^2}{F_0}}, \tag{4.11}$$

where we have defined the geodesic impact parameter b. If we now compute the coordinate angular velocity of the orbit, we find that

$$\Omega_0 \equiv \frac{\dot{\theta}}{\dot{t}} = \frac{J}{E}\frac{F_0}{r_s^2} = \sqrt{\frac{F_0}{r_s^2}} \tag{4.12}$$

where we have simply substituted the definitions of E and J and used Eqn. 4.11. Since $r_s = r_0$, we see that this is exactly exactly the coefficient of l in Eqn. 4.7, our expression for the WKB quasinormal modes.

The other coefficient also has a physical interpretation. We can perform a stability analysis of the null circular orbits by linearizing the geodesic equation $\dot{r}^2 - V_r = 0$, i.e. by substituting in a solution $r(t) = r_0 + \delta r(t)$ and solving for $\delta r(t)$. We find that

$$(\delta r'(t)\dot{t})^2 - \left[V_r(r_0) + V_r'(r_0)\delta r(t) + \frac{1}{2}V_r''(r_0)\delta r(t)^2\right] = 0, \tag{4.13}$$

and since $V(r_0) = V'(r_0) = 0$, we have

$$\delta r'(t) = \sqrt{\frac{V_r''(r_0)}{2\dot{t}^2}}\delta r(t). \tag{4.14}$$

When $V_r''(r_0)$ is positive, we see that perturbations about the unstable orbit grow exponentially in t, as

$$\delta r(t) \sim e^{\lambda t}, \tag{4.15}$$

and writing in terms of a derivative with respect to the r_* coordinate, we find that

$$\lambda = \sqrt{\frac{V_r''}{2\dot{t}^2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{r_0^2}{F_0}\frac{d^2}{dr_*^2}\left[\frac{F(r)}{r^2}\right]\bigg|_{r=r_0}}, \tag{4.16}$$

where λ is called the (principal) Lyapunov exponent. This is the other coefficient in Eqn. (4.7). In terms of Ω_0 and λ, we therefore find that

$$\omega_{\mathrm{WKB}} = l\Omega_0 - i(n + 1/2)|\lambda|. \tag{4.17}$$

This completes the proof of Thm. 8. □

We remark that photon sphere modes also exist for axisymmetric spacetimes like Kerr and Kerr-de Sitter, as studied in [4]. There, the relevant photon orbits are equatorial orbits, since those orbits will maximize the angular momentum number l.

With this result in hand, let us calculate the photon sphere modes of the RNdS spacetime. For RNdS, we have

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{L^2} \tag{4.18}$$

which gives an effective radial potential

$$V_r = E^2 - F(r)\frac{J^2}{r^2} = \frac{J^2}{b^2}\left[1 - \frac{b^2}{r^2}F(r)\right]. \tag{4.19}$$

We then impose the null circular orbit conditions $V_r(r_s) = V_r'(r_s) = 0$, which can readily be rewritten as conditions on r_s and b_s:

$$b_s = \sqrt{\frac{r_s^2}{F(r_s)}}, \quad 2F(r_s) - r_s F'(r)|_{r=r_s} = 0 \tag{4.20}$$

and substituting the explicit form of $F(r)$ yields the photon sphere radius and the corresponding impact parameter,

$$r_s = \frac{3M + \sqrt{9M^2 - 8Q^2}}{2}, \quad b_s = \frac{Lr_s^2}{\sqrt{L^2(r_s^2 - 2Mr_s + Q^2) - r_s^4}}. \tag{4.21}$$

We get the coordinate angular velocity for free,

$$\Omega_{\text{RNdS}} = \frac{1}{b_s}, \tag{4.22}$$

and calculating the principal Lyapunov exponent then gives

$$\lambda_{\text{RNdS}} = \sqrt{\frac{V_r''}{2\dot{t}^2}}\Bigg|_{r=r_s, b=b_s} = \frac{\sqrt{r_s^2 - 2Q^2}}{b_s r_s}. \tag{4.23}$$

This is all the information we need to write down the photon sphere modes for the Reissner-Nordström-de Sitter black hole: they are

$$\omega_{\text{PS}} = l\Omega_{\text{RNdS}} - i\left(n + \frac{1}{2}\right)\lambda_{\text{RNdS}} \tag{4.24}$$

with Ω_{RNds} and λ_{RNds} given by Eqns. 4.22 and 4.23 respectively.

At this point, it is worth noting that an equivalent photon sphere analysis can be performed for the Kerr-de Sitter spacetime, as was done in [4]. As it turns out, there are two notable features of the quasinormal modes of Kerr-de Sitter as compared to Reissner-Nordström-de Sitter. The first is that Kerr-de Sitter seems to have only one family of quasinormal modes, in contrast with the three families of RNdS. The only family that appears is the photon sphere modes, so that WKB methods remain reliable even near

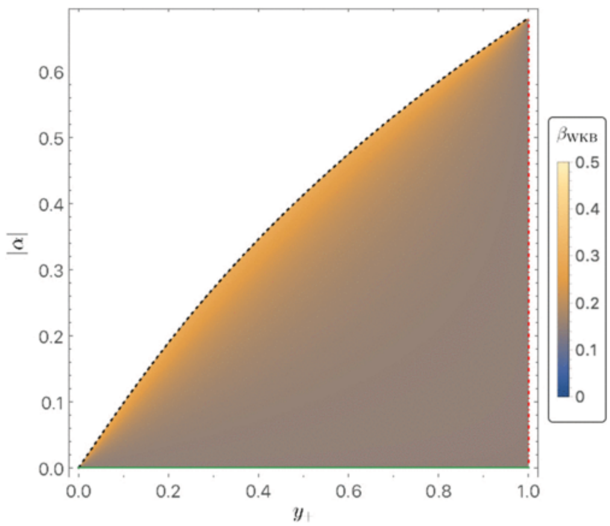

FIG. 4: The WKB approximation for β throughout the moduli space of Kerr-de Sitter, reproduced from [4]. Here, $\alpha = a/r_c$ is a normalized version of the angular momentum and $y_+ = r_+/r_c$ characterizes the size of the black hole relative to r_c. The dashed black line represents the extremal limit. For the massless scalar field, β does not exceed $1/2$ in any non-extremal Kerr-de Sitter spacetime.

extremality. This means there are no slowly decaying near-extremal modes in the Kerr-de Sitter black hole, which leads to the second difference– as we will show, the near-extremal RNdS black hole does violate strong cosmic censorship, but the near-extremal Kerr-de Sitter black hole does not.

The WKB results for Kerr-de Sitter appear in Fig. 4, reproduced from Dias et al. in [4]. The interpretation of this result is as follows– since no new family of quasinormal modes emerges near extremality, all (scalar field) quasinormal modes of the Kerr-de Sitter spacetime are found to decay slowly enough near the Cauchy horizon, so that the Christodoulou formulation of strong cosmic censorship is respected. Ref. [4] also found that the numerical results for Kerr-de Sitter support the photon sphere mode calculations, with the error in β decreasing towards extremality.

Moreover, these results are also valid for *metric perturbations* to Kerr-de Sitter, since the photon sphere modes are effectively independent of spin, which led Dias et al. to conclude that not only do scalar field quasinormal modes obey strong cosmic censorship, but more importantly, *linearized gravitational perturbations of the non-extremal Kerr-de Sitter spacetime also uphold strong cosmic censorship.*

In contrast, for the Reissner-Nordström-de Sitter spacetime, β continues to increase beyond the critical value of 1/2 near extremality, as was found in [22], and in fact a new family of near-extremal modes eventually takes over as the slowest-decaying quasinormal modes. It is these near-extremal modes which ultimately determine the severity of the violation of strong cosmic censorship in the RNdS black hole.

4.3. de Sitter modes

For completeness, we mention the existence of a family of quasinormal modes which dominates in the limit of a small black hole ($y_+ = r_+/r_c \ll 1$). These modes are deformations of the quasinormal modes of pure de Sitter space. Numerically, they do not seem to be of much importance to the Kerr-de Sitter spacetime, which is well-described by the photon sphere modes alone. For the massless scalar field, these de Sitter modes were computed explicitly to be [13, 22]

$$\omega_{n=0,\mathrm{dS}} = -il\kappa_c^{\mathrm{dS}} \tag{4.25a}$$

$$\omega_{n\neq 0,\mathrm{dS}} = -i\kappa_c^{\mathrm{dS}}(n+l+1), \tag{4.25b}$$

where κ_c^{dS} is the surface gravity of the cosmological horizon in pure de Sitter space and the $n=0, l=1$ mode dominates.

We shall see these again in our discussion of numerics, but they do not otherwise hold too much interest for our problem. However, it is worth noting that Cardoso et al. numerically demonstrated the existence of both photon sphere and de Sitter modes of the RNdS spacetime for which $\beta > 1/2$ in a region of parameter space somewhat near extremality but before the near-extremal modes take over [22]. Thus even small black holes close enough to extremality may violate the Christodoulou formulation of strong cosmic censorship.

4.4. Near-extremal modes

Of the three families of quasinormal modes, the near-extremal modes are perhaps the most critical. Near extremality, the blueshift instability is weakest, and it is in this regime that we expect to observe the most severe violations of strong cosmic censorship. To study these modes analytically, we consider the near-horizon geometry in the near-extremal limit as $r_- \to r_+$.

Our main result in this limit is the following theorem [23].

Theorem 9. *The quasinormal mode frequencies of the near-extremal Reissner-Nordström-de Sitter black hole for massless scalar field perturbations are given by*

$$\omega_{\rm NE} = -i\kappa_+(n+l+1), \tag{4.26}$$

where κ_+ is the surface gravity of the event horizon and n, l are non-negative integers.

Following [23], let us first consider the behavior of the massless scalar field in the near-horizon region of the asymptotically flat, near-extremal Reissner-Nordström background, i.e. where $\Lambda = 0$ and we expand in the dimensionless parameters r_0 and ρ defined by

$$M = Q(1+r_0), \quad r = Q(1+\rho).$$

The limits $r_0 \to 0$ and $\rho \to 0$ correspond to the near-extremal and near-horizon limits, respectively. Our aim will be to zoom in on the near-horizon region and analytically solve the wave equation in this region, deriving constraints on the quasinormal mode frequencies in the process.

The asymptotically flat Reissner-Nordström solution takes the form

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2 d\Omega^2, \tag{4.27}$$

with $F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$. If we expand $F(r)$ to leading order in r_0 and ρ, we find that

$$\begin{aligned} F(r) &= \rho^2 - 2r_0 \\ &= \rho^2 - 2(M-Q)/Q. \end{aligned} \tag{4.28}$$

If we now define $\tilde{\rho} \equiv Q\rho$ and $\rho_0 \equiv \sqrt{2Q(M-Q)}$, we can then compactly rewrite the Reissner-Nordström metric in the near-extremal, near-horizon limit as

$$ds^2 = -\frac{\tilde{\rho}^2 - \rho_0^2}{Q^2}dt^2 + \frac{Q^2}{\tilde{\rho}^2 - \rho_0^2}d\tilde{\rho}^2 + Q^2 d\Omega^2. \tag{4.29}$$

Using this form of the metric, we can write the radial QNM equation, Eqn. 3.5, in terms of the near-horizon coordinate $\tilde{\rho}$ so that our quasinormal modes solve

$$\frac{d^2\psi}{d\rho_*^2} + (\omega^2 - V_l)\psi = 0, \tag{4.30}$$

with the potential

$$V_l \equiv \frac{l(l+1)(\tilde{\rho}^2 - \rho_0^2)}{Q^4} \tag{4.31}$$

and the tortoise coordinate ρ_* defined by

$$d\rho_* = \frac{d\tilde{\rho}}{\left(\frac{\tilde{\rho}^2-\rho_0^2}{Q^2}\right)} \implies \rho_* = \frac{1}{2\kappa_+}\ln\left(\frac{\tilde{\rho}-\rho_0}{\tilde{\rho}+\rho_0}\right), \tag{4.32}$$

where $\kappa_+ = \rho_0/Q^2$ is the surface gravity near the event horizon. By a change of variables, Eqn. 4.30 can be written as [23]

$$x(1-x)\frac{d^2}{dx^2}\psi + \left(1-\frac{3}{2}x\right)\frac{d}{dx}\psi + \left(\frac{\omega^2}{4\kappa_+^2 x} - \frac{l(l+1)}{4(1-x)}\right)\psi = 0 \tag{4.33}$$

in terms of $x \equiv \frac{1}{\cosh^2(\kappa_+\rho_*)}$. The solution to this equation can be written in terms of a standard hypergeometric function,

$$\psi = x^{-i\omega/2\kappa_+}(1-x)^{l/2}{}_2F_1[a,b,c;x] \tag{4.34}$$

where

$$a = -\frac{i\omega}{2\kappa_+} - \frac{l}{2} + \frac{1}{2}, \quad b = -\frac{i\omega}{2\kappa_+} - \frac{l}{2}, \quad c = 1 - \frac{i\omega}{\kappa_+}. \tag{4.35}$$

To compute the QNM frequencies, we must now impose the QNM boundary conditions. Here, we take Dirichlet boundary conditions and require that ψ is not just outgoing but in fact vanishes as $\tilde{\rho} \to \infty$, which corresponds to the limits $\rho_* \to 0, x \to 1$. Physically, these boundary conditions are motivated by the idea that the near-extremal modes are highly localized to near the event horizon, and should therefore vanish at spatial infinity. In this limit, note that ${}_2F_1$ takes on a simple form:

$$\lim_{x\to 1} {}_2F_1(a,b,c;x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4.36}$$

This vanishes whenever the arguments of the gamma functions in the denominator take non-positive integer values, i.e.

$$c-a=-n \quad \text{or} \quad c-b=-n \tag{4.37}$$

with $n = 0,1,2,\ldots$. Using the definitions of a,b,c in 4.35, this condition yields the near-extremal quasinormal mode spectra

$$\omega = -i\kappa_+(2n+l+1), \quad \omega = -i\kappa_+(2n+l+2). \tag{4.38}$$

Noting that $2n+1$ covers all positive odd integers starting from 1 and $2n+2$ includes all positive even integers starting from 2, we can combine these expressions to arrive at the final condition that

$$\omega_{\rm NE} = -i\kappa_+(n+l+1), \tag{4.39}$$

with $n, l \in \mathbb{Z}_{\geq 0}$ as originally stated in Thm. 9. While this result was derived for the asymptotically flat Reissner-Nordström spacetime, it nevertheless seems to agree well with numerics in the RNdS spacetime, as is demonstrated in [22]. This is because the near-extremal modes are highly localized around the event horizon, and therefore do not depend strongly on the full geometry in the far region.

In fact, this is not the end of the story. To find the slowest-decaying mode, let us set $n = 0$ and $l = 0$ in Eqn. 4.39. If we take the extremal limit, $\kappa_+ \to \kappa_-$, we see that $\beta \to 1$ for the massless scalar field in the near-extremal RNdS spacetime. Thus the massless scalar field violates the Christodoulou formulation of strong cosmic censorship (which requires $\beta < 1/2$) but not the C^1 formulation, thanks to the near-extremal modes.

Furthermore, this property of the near-extremal modes upholding the C^1 formulation turns out to be specific to the *massless* scalar field. Following on the numerical work of Cardoso et al, Ref. [24] studied the behavior of the massive (charged) scalar field in the near-extremal RNdS spacetime, exposing a much worse violation of strong cosmic censorship. Setting the scalar charge to zero in their result for the near-extremal modes, we have

$$\omega_{\mathrm{NE}} = -i\kappa_- \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \hat{\eta}} \right), \tag{4.40}$$

where $\hat{\eta}$ is real and positive and given in terms of l and y_+ as

$$\hat{\eta} \equiv \Xi \big[y_+^2 \tilde{\mu}^2 + l(l+1) \big] \tag{4.41a}$$

$$\Xi = \frac{1 + 2y_+ + 3y_+^2}{(1 - y_+)(1 + 3y_+)}, \tag{4.41b}$$

and $\tilde{\mu} = \mu r_c$ is the scalar mass in units of r_c.

In the $\tilde{\mu} = 0$ case, this formula just reproduces our previous result– the slowest-decaying mode has $l = 0 \implies \hat{\eta} = 0$, so that $\omega_{\mathrm{NE}} \to -i\kappa_-$. But for a nonzero mass, we notice that even for the slowest-decaying $n = 0, l = 0$ modes, as $y_+ \to 1$ (the limit in which the event horizon and cosmological horizon coincide), Ξ diverges (as can be read off from Eqn. 4.41b) and thus $\hat{\eta}$ grows arbitrarily large. In particular, this means that β can take on values exceeding 1, so it is not in general true that the near-extremal modes uphold the C^1 formulation of strong cosmic censorship.

The analogous calculation for the near-extremal modes for the gravitoelectromagnetic perturbation reveals similar behavior to the massive scalar case. As shown in [15], the gravitoelectromagnetic quasinormal modes in the near-extremal limit have a modified spectrum,

$$\omega_{\mathrm{NE}} = -i\kappa_- \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \hat{\eta}_-} \right), \tag{4.42}$$

where now $\hat{\eta}_-$ is given by

$$\hat{\eta}_- = 1 + \Xi l(l+1) - \sqrt{[1 + \Xi l(l+1)]^2 - \Xi^2 (l+2)(l+1)l(l-1)}, \tag{4.43}$$

with Ξ defined in Eqn. 4.41b and $n = 0, l = 2$ as the slowest-decaying mode. To understand this formula, let us consider a small black hole, where r_c is large compared to r_+ and hence $y_+ \to 0$. In this limit, $\Xi \to 1$ so that $\hat{\eta}_- \to 2$ and then

$$\beta = -\frac{\mathrm{Im}(\omega_{\mathrm{NE}})}{\kappa_-} \to \left(\frac{1}{2} + \sqrt{\frac{1}{4} + 2} \right) = 2. \tag{4.44}$$

This is a striking result. Since Ξ is an increasing function of y_+, it follows that even for small RNdS black holes close to extremality, $\beta > 2$, which violates not only the Christodoulou formulation but the C^2 formulation of strong cosmic censorship. Conversely, if we take the limit of $y_+ \to 1$, then just as we saw in the massive scalar case, Ξ blows up, which means β can reach arbitrarily large values. Thus by Thm. 7, linear gravitoelectromagnetic perturbations can be extended as smoothly as we like across the Cauchy horizon, provided that we take a sufficiently large black hole near extremality. We conclude that the Christodoulou formulation and all C^r (“smooth”) formulations of strong cosmic censorship are violated by the Reissner-Nordström-de Sitter family of black holes.

These results are numerically confirmed and summarized in Fig. 5. In this figure, all three families of gravitoelectromagnetic quasinormal modes are shown. We see that the Christodoulou formulation is violated for significant regions of moduli space, and in the upper right (corresponding to large, near-extremal black holes where y_+ and Q/Q_{ext} are both close to 1), values of β are calculated which are well above 2 (i.e. the cutoff for the C^2 formulation, which was previously shown to be true in asymptotically flat space). Based on our analytic calculations, we expect this trend to hold for larger black holes near extremality.

5. CONCLUSIONS

In this work, we have discussed a number of different formulations of Penrose's strong cosmic censorship conjecture. We have presented an argument in support of the

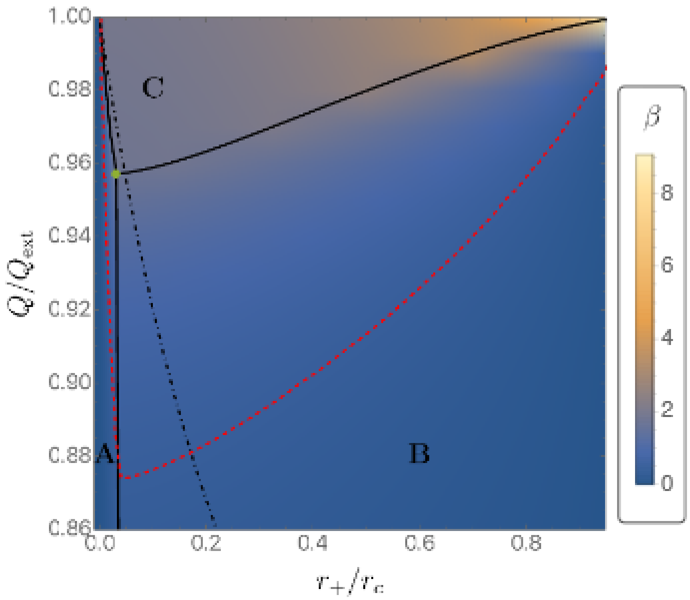

FIG. 5: An illustration of the values of β for the three families of quasinormal modes corresponding to gravitoelectromagnetic perturbations to the RNdS black hole, reproduced from [15]. Region A is the de Sitter modes (at low values of $y_+ = r_+/r_c$), region B is the photon sphere modes (for intermediate and large values of y_+ with Q still far from extremality), and region C is the near-extremal modes (for Q near extremality, as the name indicates). The red dashed line indicates the region of moduli space above which the Christodoulou formulation of strong cosmic censorship is violated, i.e. $\beta > 1/2$.

Christodoulou formulation of strong cosmic censorship as the most physically relevant formulation when it comes to the breakdown of the field equations at the Cauchy horizon, and rederived a result from [15, 16] establishing that the quasinormal mode frequencies and in particular the spectral gap are critical to the decay of linear perturbations near the Cauchy horizon.

Having shown that a knowledge of the slowest-decaying quasinormal mode frequencies is sufficient to establish a violation of the Christodoulou formulation of strong cosmic censorship, we then considered the specific cases of the Reissner-Nordström-de Sitter and Kerr-de Sitter spacetimes and calculated the QNM frequencies analytically for the massless scalar field, citing analogous results for other perturbations where appropriate. Our key results are drawn from [22], who showed that the massless scalar field in the near-extremal Reissner-Nordström-de Sitter spacetime violates the Christodoulou formulation but not the C^1 formulation of strong cosmic censorship; from [4], who showed that no equivalent violations of strong cosmic censorship are observed for the Kerr-de Sitter spacetime for the scalar field and the (linear) metric perturbation itself; and from [15], who showed that gravitoelectromagnetic perturbations in the near-extremal Reissner-Nordström-de Sitter spacetime violate all smooth formulations of strong cosmic censorship.

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